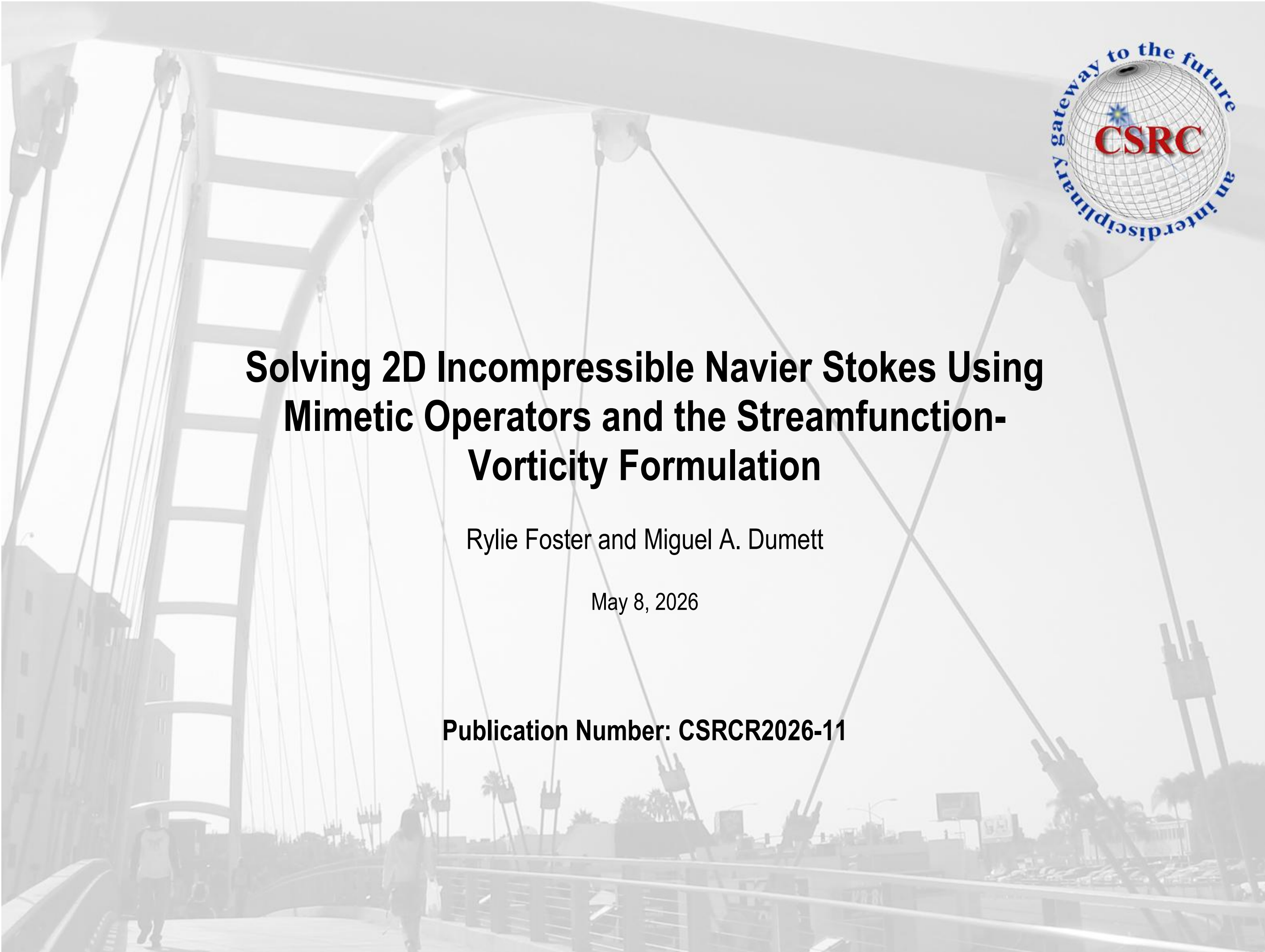

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Solving 2D Incompressible Navier Stokes Using Mimetic Operators and the Streamfunction-Vorticity Formulation

Rylie Foster[*] Miguel A. Dumett[‡]

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Abstract

Mimetic Operators provided by MOLE were used to solve the 2D incompressible Navier-Stokes equations on a square domain with a periodic boundary condition. In order to increase compatibility with MOLE's staggered grid and scalar/vector spatial separation, the streamfunction-vorticity formulation of the Navier-Stokes equation was used. For validation, simulated solutions were compared to the Taylor-Green vortex analytical solution for three separate viscosity conditions, three final times, and 2nd, 4th, and 6th order mimetic operators. Simulated results indicated an order-of-magnitude improvement from 2nd-order to fourth-order operators, and little improvement from 4th-order to 6th-order operators.

1 Introduction

Fluid simulations are critical to the development of a wide variety of fields, from the transportation industry, environmental research, to even the development of modern medical technologies. As the demand for fluid models is both great and varied, there is demand for a wide variety of capabilities in fluid models, with some use cases emphasizing accuracy, some requiring speed, and many others requiring the ability to recreate some specific feature of real-world flows. In this paper, we model a two-dimensional incompressible fluid flow on a square domain with a periodic boundary condition. We use the streamfunction-vorticity formulation of the Navier-Stokes equations, and our solver uses mimetic operators provided by the MOLE library (Mimetic Operators Library Enhanced)[CC20, CDC24]. Algorithms for solving the Navier-Stokes equations face two key difficulties. The first challenge is that both the velocity field and pressure field of the solution must be calculated, a requirement which is generally satisfied by advancing the velocity field in time, and then calculating a correction to satisfy the pressure constraint by solving a Poisson problem. Our algorithm also requires that a Poisson problem be solved at each time step, which we achieve by inverting a mimetic divergence operator. The second challenge is that algorithms for solving Navier-Stokes on a singular spatial discretization are generally unstable. This is usually resolved by introducing offset meshes for the horizontal and vertical components of the field, making the solver a finite-volume scheme which respects the divergence-free condition of an incompressible fluid flow. This solution conflicts with the geometry of a mimetic solver, which already incorporates offset meshes for scalars and vectors. Using the streamfunction-vorticity formulation of Navier-Stokes, which substitutes the vector and scalar-valued dependent variables $\vec{v}$ and p in the original equations for scalar-valued dependent variables ω and ψ, respectively, the vorticity of the velocity field and the streamfunction of the velocity field, allows us to sidestep the issue.

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2 Mathematical Formulation

2.1 Navier-Stokes Equations

The incompressible Navier-Stokes equations describe the evolution of the velocity flow-field of an incompressible fluid. The traditional formulation of the Navier-Stokes equations is

$$\frac{\partial \vec{v}}{\partial t} + \nabla \cdot (\vec{v} \otimes \vec{v}) + \nabla p = \nu \nabla^2 \vec{v} \tag{1}$$

$$\nabla \cdot \vec{v} = 0, \tag{2}$$

where $\vec{v} = (u, v)$ is a vector describing the velocity of flow at a point in the domain, p is the pressure of the fluid, and $\nu = \frac{1}{Re}$, the viscosity of the fluid. The first equation, which is often called the momentum equation, summarizes the forces acting on a point of the fluid flow. The second equation imposes a divergence-free condition on the fluid, representing the fact that our fluid is incompressible and that at no point in the domain is fluid being added to the system.

2.2 Streamfunction-Vorticity Formulation

To reformulate the Navier-Stokes equations into an all-scalar system, we introduce new variables $\omega(x, y) = \frac{\partial v(x,y)}{\partial x} - \frac{\partial u(x,y)}{\partial y} = \nabla \times \vec{v}$ and $\psi(x, y)$, where $\psi(x, y)$ is defined such that $\frac{\partial \psi(x,y)}{\partial x} = -v(x, y)$ and $\frac{\partial \psi(x,y)}{\partial y} = u(x, y)$. We begin by reformulating (1), writing

$$\begin{aligned} \nabla \times \frac{\partial \vec{v}}{\partial t} &= \nabla \times (-\nabla \cdot (\vec{v} \otimes \vec{v}) - \nabla p + \nu \nabla^2 \vec{v}) \\ \frac{\partial \omega}{\partial t} &= \nabla \times (-\nabla \cdot (\vec{v} \otimes \vec{v})) + \nu \nabla^2 \omega \\ \frac{\partial \omega}{\partial t} &= -\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} + \nu \nabla^2 \omega \end{aligned} \tag{3}$$

We defer to citing [JH15] for final step of (3), as the calculation of the tensor-product curl is lengthy.

Next, we observe

$$\begin{aligned} \nabla^2 \psi &= \frac{\partial}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \psi}{\partial y} \\ &= -\frac{\partial}{\partial x} v + \frac{\partial}{\partial y} u \\ &= -\omega. \end{aligned} \tag{4}$$

Equations (3) and (4) are the streamfunction-vorticity equations. They are equivalent to Navier-Stokes, but they relate strictly scalar quantities on their domain.

2.3 Taylor-Green Vortex

The solver was tested by recreating the Taylor-Green vortex analytical solution to Navier-Stokes, a benchmark which has been recommended for 2D incompressible fluid solvers [SJC26, HSL+21], and which has an analytical solution. The problem is defined on a square box with a periodic boundary condition. Altogether, the domain is $\Omega \times [0, T] = [0, 1]^2 \times [0, T]$, with final times $T = 0.5, 1$, and 2. The initial condition is given by

$$\begin{aligned} u(x, y, 0) &= \cos(2\pi x)\sin(2\pi y) \\ v(x, y, 0) &= -\sin(2\pi x)\cos(2\pi y) \\ p(x, y, 0) &= -\frac{1}{4}(\cos(4\pi x) + \cos(4\pi y)). \end{aligned}$$

The exact solution is given by

$$u(x, y, t) = \cos(2\pi x)\sin(2\pi y)e^{-8\nu\pi^2 t}$$
$$v(x, y, t) = -\sin(2\pi x)\cos(2\pi y)e^{-8\nu\pi^2 t}$$
$$p(x, y, t) = -\frac{1}{4}(\cos(4\pi x) + \cos(4\pi y))e^{-16\nu\pi^2 t}.$$

3 Design and Implementation

3.1 Algorithm Outline

Our simulation scheme is explicit in time, with fixed time steps $\Delta t = 0.001$. Without particular consideration for the underlying solver, our algorithm will take the following form:

1. Given a Navier-Stokes initial condition $(u_0(x, y), v_0(x, y))$, calculate $\psi_x = -v$, $\psi_y = u$, $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$
2. for $t \in \{t_k\}_{t_1}^{t_N}$:
   (a) calculate $\frac{\Delta\omega^k}{\Delta t}$ using (3),
   (b) calculate $\omega^{k+1} = \omega^k + \Delta t \frac{\Delta\omega^k}{\Delta t}$,
   (c) calculate ψ^{k+1} by solving the Poisson problem in (4),
   (d) calculate $\frac{\Delta\psi}{\Delta x}^{k+1}$, $\frac{\Delta\psi}{\Delta y}^{k+1}$ from ψ^{k+1},
3. obtain u^N, v^N from ψ^N.

Implementing this algorithm is now simply a problem of formulating it on the offset mesh used by MOLE and gathering the required mimetic operators.

3.2 Required Operators

Before we can implement the above algorithm in MOLE, we must note that MOLE requires separate face and center meshes for scalar and vector quantities respectively. While the the streamfunction and vorticity variables are both scalars, the gradients of those variables, which appear in (3), are vectors and need to be projected onto the scalar mesh to perform step (a) in our algorithm. To handle this, we use interpolators which are provided by MOLE. Altogether, our solver utilizes the following operators from MOLE:

- two interpolation operators I_x and I_y which map the x, and y components of vectors from the face mesh to the center mesh, constructed with MOLE's interpolFacesToCentersG1DPeriodic function
- two operators G_x and G_y for the x and y components of the mimetic gradient operator, used to approximate terms in (3), constructed by first calling MOLE's grad2D function, and then extracting the sub-matrices corresponding to the x and y components of the projection,
- a mimetic Laplacian L, to approximate the left-hand side of (4), constructed by calling MOLE's lap2D function.

As we can see, each of these operators is either readily available in MOLE, or trivial to construct from existing MOLE functions.

3.3 Mimetic Solver

Given the mimetic operators listed above, as well as initial conditions u_0 and v_0 of the velocity field, we can now describe our mimetic algorithm:

1. Let $\psi_{x0} = -v_0$, $\psi_{y0} = u_0$, $\omega_0 = \frac{\Delta v}{\Delta x}|_0 - \frac{\Delta u}{\Delta y}|_0$
2. for $k = 1, ..., N$:
   (a) Let $A^k = -\psi_y^k G_x \omega^k + \psi_x^k G_y \omega^k$,
   (b) $\omega^{k+1} = (I + \Delta t A)\omega^k$,
   (c) $\psi^{k+1} = -L \backslash \omega^{k+1}$,
   (d) $\frac{\Delta \psi}{\Delta x}^{k+1} = G_x \psi^{k+1}$, $\frac{\Delta \psi}{\Delta y}^{k+1} = G_y \psi^{k+1}$,
3. $u^N = \psi_{y0}$, $v^N = -\psi_{x0}$.

The final velocity flow field is now stored in u^N and v^N.

4 Results

Numerical simulations were carried out to final times $T = 0.5, 1$, and 2, and with fluid viscosities $\nu = 0.01, 0.0025$, and 0.001. The spatial resolution used was $\Delta x \approx 0.01961$, and the time step used was $dt = 0.001$. Simulated final flow field components were compared to the analytic results and the L_∞ error and maximum relative error were recorded. The highest relative error observed was 0.18507729%. Plots of the final flow field and tabulated errors are below. The maximum divergence was also monitored during a number of Taylor-Green vortex simulations.

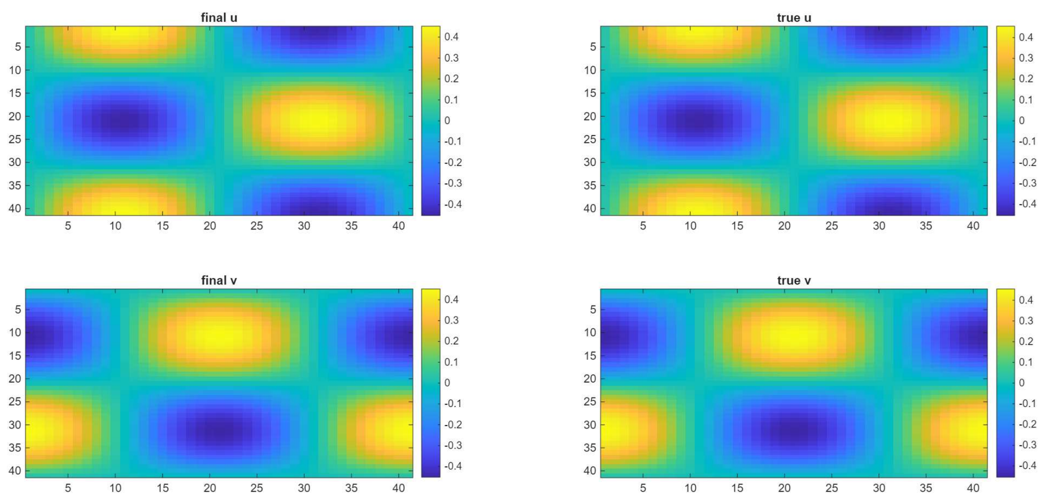

Figure 1: Simulated and analytic solutions for u and v. $\nu = 0.01$, $T = 1$.

2nd Order - Maximum and Relative Error, u component			
	$T = 0.5$	$T = 1$	$T = 2$
$\nu = 0.01$	Max: 1.79e-03, Rel: 1.33e-03	Max: 1.05e-03, Rel: 1.16e-03	Max: 3.36e-04, Rel: 8.16e-04
$\nu = 0.0025$	Max:3.15e-03, Rel: 1.74e-03	Max: 2.76e-03, Rel: 1.68e-03	Max: 2.11e-03, Rel: 1.57e-03
$\nu = 0.001$	Max: 3.52e-03, Rel: 1.83e-03	Max: 3.34e-03, Rel: 1.81e-03	Max: 3.00e-03, Rel: 1.76e-03

2nd Order - Maximum and Relative Error, v component			
	$T = 0.5$	$T = 1$	$T = 2$
$\nu = 0.01$	Max: 1.79e-03, Rel: 1.33e-03	Max: 1.05e-03, Rel: 1.16e-03	Max: 3.36e-04, Rel: 8.16e-04
$\nu = 0.0025$	Max: 3.15e-03, Rel: 1.74e-03	Max: 2.76e-03, Rel: 1.68e-03	Max: 2.11e-03, Rel: 1.57e-03
$\nu = 0.001$	Max: 3.52e-03, Rel: 1.83e-03	Max: 3.34e-03, Rel: 1.81e-03	Max: 3.00e-03, Rel: 1.76e-03

4th Order - Maximum and Relative Error, u component			
	$T = 0.5$	$T = 1$	$T = 2$
$\nu = 0.01$	Max: 4.20e-04, Rel: 3.12e-04	Max: 2.13e-04, Rel: 2.35e-04	Max: 3.27e-05, Rel: 7.94e-05
$\nu = 0.0025$	Max:1.60e-04, Rel: 8.86e-05	Max: 1.37e-04, Rel: 8.38e-05	Max: 1.00e-04, Rel: 7.43e-05
$\nu = 0.001$	Max: 6.41e-05, Rel: 3.34e-05	Max: 3.34e-03, Rel:0.18%	Max: 5.32e-05, Rel: 3.11e-05

4th Order - Maximum and Relative Error, v component			
	$T = 0.5$	$T = 1$	$T = 2$
$\nu = 0.01$	Max: 4.20e-04, Rel: 3.12e-04	Max: 2.13e-04, Rel:2.35e-04	Max: 3.27e-05, Rel: 7.94e-05
$\nu = 0.0025$	Max: 1.60e-04, Rel: 8.86e-05	Max: 1.37e-04, Rel: 8.38e-05	Max: 2.11e-03, Rel: 7.43e-05
$\nu = 0.001$	Max: 6.41e-05, Rel: 3.34e-05	Max: 6.02e-05, Rel: 3.26e-05	Max: 5.32e-05, Rel: 3.11e-05

6th Order - Maximum and Relative Error, u component			
	$T = 0.5$	$T = 1$	$T = 2$
$\nu = 0.01$	Max: 4.27e-04, Rel: 3.17e-04	Max: 2.17e-04, Rel: 2.39e-04	Max: 3.42e-05, Rel: 8.31e-05
$\nu = 0.0025$	Max:1.70e-04, Rel: 9.38e-05	Max: 1.46e-04, Rel: 8.90e-05	Max: 1.07e-04, Rel: 7.92e-05
$\nu = 0.001$	Max: 7.43e-05, Rel: 3.87e-05	Max: 7.00e-05, Rel: 3.79e-05	Max: 6.20e-05, Rel: 3.63e-05

6th Order - Maximum and Relative Error, v component			
	$T = 0.5$	$T = 1$	$T = 2$
$\nu = 0.01$	Max: 4.27e-04, Rel: 3.17e-04	Max: 2.17e-04, Rel: 2.39e-04	Max: 3.42e-05, Rel: 8.31e-05
$\nu = 0.0025$	Max: 1.70e-04, Rel: 9.38e-05	Max: 1.46e-04, Rel: 8.90e-05	Max: 1.07e-04, Rel: 7.92e-05
$\nu = 0.001$	Max: 7.43e-05, Rel: 3.87e-05	Max: 7.00e-05, Rel: 3.79e-05	Max: 6.20e-05, Rel: 3.63e-05

Figure 2: Maximum and relative error on the domain at time T.

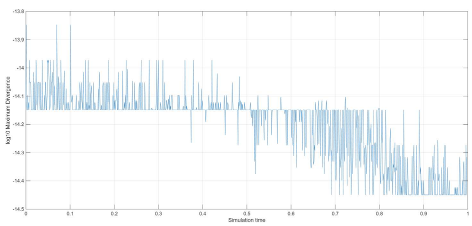

Figure 3: Maximum field divergence with 4th-order operators, $\nu = 0.01$.

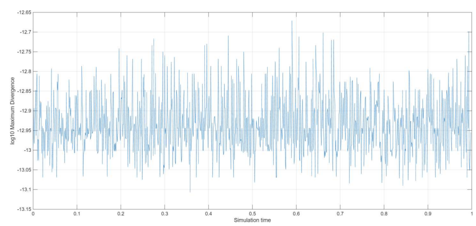

Figure 4: Maximum field divergence with 4th-order operators, $\nu = 0.0025$

5 Conclusion

The mimetic solver recreated the overall shape of the analytic solution without issue. From the error data, we observe an order-of magnitude in maximum and relative errors from 2nd to 4th-order operators. We observe little improvement from 4th to 6th-order operators. Our mimetic solution also appears to be stable, as relative errors do not grow as we increase T. For the conditions tested, the divergence-free condition was held on the order of 10^{-12}.

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