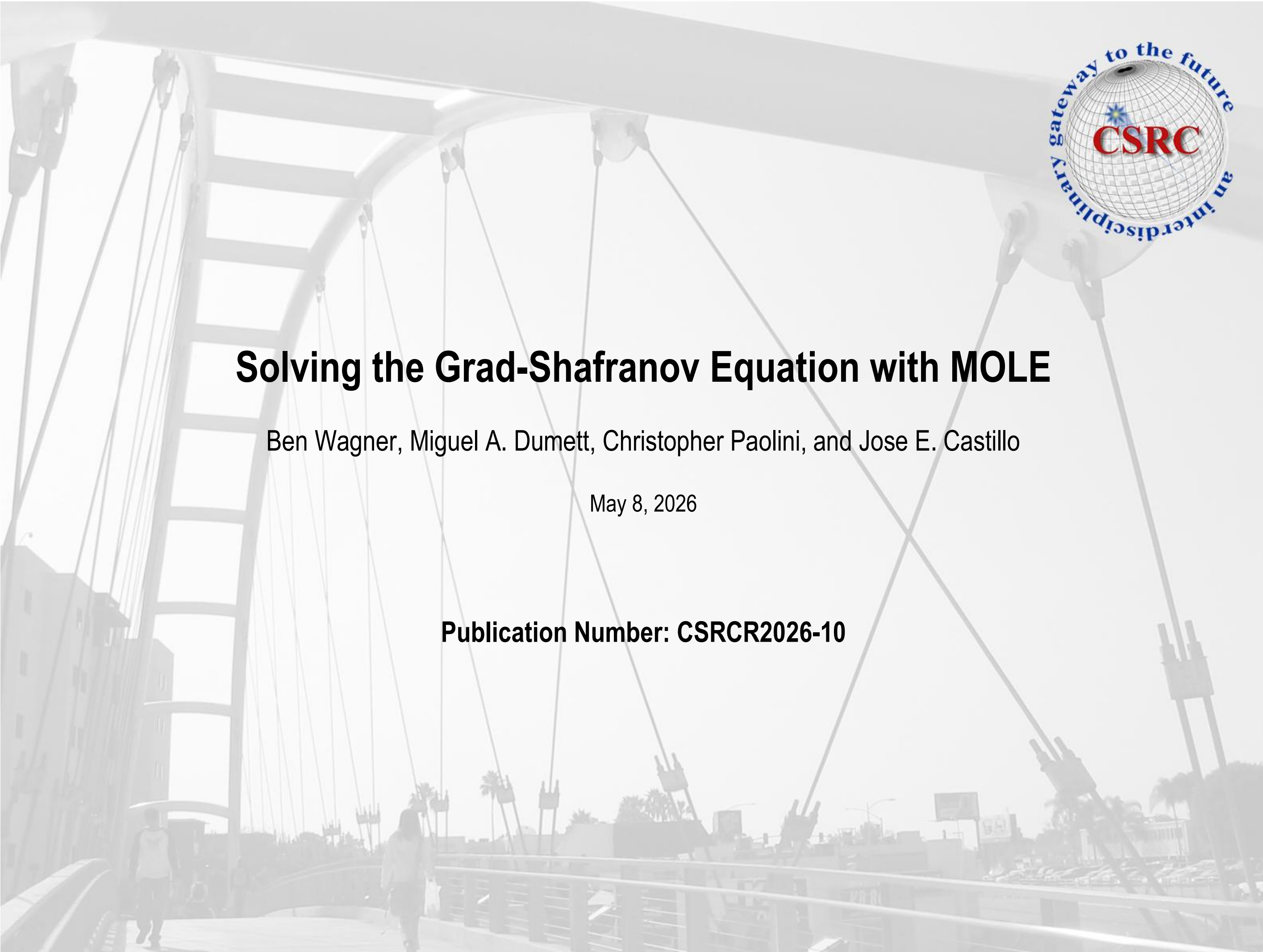

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Computational Science Research Center
College of Sciences
5500 Campanile Drive
San Diego, CA 92182-1245
(619) 594-3430

Solving the Grad-Shafranov Equation with MOLE

Ben Wagner[1], Miguel A. Dumett[2], Christopher Paolini[2], and Jose E. Castillo[2]

[1]Computational Science PhD Program at San Diego State University
[2]Computational Science Research Center at San Diego State University
{*bwagner4166@sdsu.edu, mdumett@sdsu.edu, paolini@engineering.sdsu.edu, jcastillo.sdsu.edu*}

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Abstract

The Grad–Shafranov (GS) equation describes the equilibrium of ideal axisymmetric toroidal plasmas and provides the initial conditions for time-dependent magnetohydrodynamic simulations used to study plasma behavior in nuclear fusion devices such as tokamaks. Current approaches solve the GS equation on one computational domain and then interpolate the solution onto another for time-dependent simulations, introducing potential numerical error. This work uses the high-order mimetic operators of the MOLE library (discrete analogs of vector calculus operators) to solve both the GS equation and the time-dependent system on a single domain, eliminating the need for interpolation and maintaining consistent high-order accuracy throughout the simulation.

1 Introduction

The Grad-Shafranov equation describes the ideal magneto-hydrodynamical equilibrium of a toroidally axisymmetric plasma. Its solution is used as an initial condition for time dependent magneto-hydrodynamic simulations in nuclear fusion devices such as tokamaks. There are Grad-Shafranov solvers that compute the equilibrium on one computational domain but then interpolate onto another for time dependent simulations, which introduces possible numerical error. The goal of this work is to use MOLE [3] to solve the Grad-Shafranov equation and eventually time dependent simulations on a single mesh.

2 Grad-Shafranov Equation

The Grad-Shafranov equation is expressed as follows:

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = \begin{cases} -\mu_0 r^2 \frac{dp}{d\psi} - F\frac{dF}{d\psi} & \text{in the plasma} \\ 0 & \text{in the vacuum} \end{cases}$$

where ψ is the poloidal magnetic flux per radian, p is the plasma pressure, and F is the poloidal current. The plasma domain is defined as the interior of the last closed flux surface, while the vacuum domain is defined as the space between the plasma boundary and the tokamak boundary. Plasma, however, is not the only current source that contributes to ψ. There are a set of polodial field coils around the tokamak which produce magnetic fields that contribute to the total flux. Their contributions are as follows:

$$\begin{aligned}
\vec{B}(\vec{x}) &= \nabla \times \vec{A}(\vec{x}) \\
&= \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\phi} + \frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)\hat{z} \\
&= -\frac{\partial A_\phi}{\partial z}\hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\phi} + \frac{1}{r}\frac{\partial (rA_\phi)}{\partial r}\hat{z} \\
\psi(\vec{x}) &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^r r(\vec{B}(\vec{x})\cdot\hat{z})drd\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^r rB_z(\vec{x})drd\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^r r(\frac{1}{r}\frac{\partial (rA_\phi)}{\partial r})drd\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^r \frac{\partial (rA_\phi)}{\partial r}drd\phi \\
&= \int_0^r \frac{\partial (rA_\phi)}{\partial r}dr \\
&= rA_\phi(\vec{x}) \\
\vec{A}(\vec{x}) &= \frac{\mu_0}{4\pi}\iiint_V \frac{\vec{J}(\vec{x}')}{\|\vec{x}-\vec{x}'\|}dV' \\
A_\phi(\vec{x}) &= \frac{\mu_0}{4\pi}\iiint_V \frac{nI}{wh}\frac{\cos\phi'}{\|\vec{x}-\vec{x}'\|}dV'
\end{aligned}$$

where $\vec{B}$ is the magnetic field, $\vec{A}$ is the magnetic vector potential, V is the volume of the coil, I is the current through the coil, n is the number of turns of the coil, w is the width of the coil, h is the height of the coil, $\vec{x}$ is the position vector from the origin to a point of interest, and $\vec{x}'$ is a position vector from the origin to a current source.

Thus, the total poloidal magnetic flux per radian can be written as a sum of the flux from the plasma and the flux from the coils:

$$\psi = \psi_P + \psi_C$$

2.1 EFIT

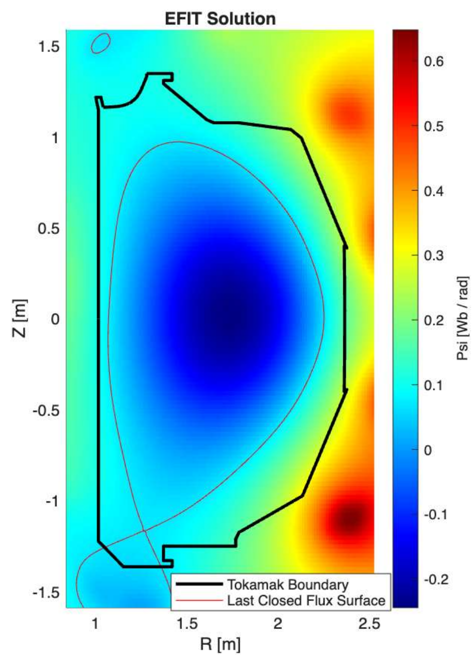

Figure 1: EFIT's solution to the Grad-Shafranov Equation

EFIT is a commonly used Grad-Shafranov solver that computes ψ in a domain that exceeds the limits of the tokamak [4]. It does so by reconstructing magnetic data recorded by sensors surrounding the tokamak and fitting coefficients to a set of basis functions that parameterize the non-linear components of the Grad-Shafranov equation, $\frac{dp}{d\psi}$ and $F\frac{dF}{d\psi}$. In its output, it provides its solution of ψ, $\frac{dp}{d\psi}$ and $F\frac{dF}{d\psi}$ as radial vectors from the magnetic axis (the elliptic fixed point in the plasma) to the plasma boundary, the (r, z) coordinates of the magnetic axis, the (r, z) coordinates of the plasma boundary, and the value of ψ on the plasma boundary. These values can be used to eliminate the non-linearity of the Grad-Shafranov equation and to compare MOLE's solution.

3 Methods

3.1 Mesh Generation

Since ψ is known on a closed loop inside of the tokamak, the tokamak domain can be decomposed into two subdomains: the plasma and the vacuum. Neither the plasma domain nor the vacuum domain are uniform, so reference grids are employed to generate their meshes [1]. Two uncoupled partial differential equations can be solved to then generate the mesh coordinates:

$$\frac{1}{\alpha^2}x_{\xi\xi} + \frac{1}{\beta^2}x_{\eta\eta} = 0$$

$$\frac{1}{\alpha^2}y_{\xi\xi} + \frac{1}{\beta^2}y_{\eta\eta} = 0$$

where α is the horizontal spacing of the reference grid and β is the vertical spacing of the reference grid. For the plasma mesh, the boundary is discretized into four segments for the left, right, bottom, and top boundaries, which are used as the boundary conditions for the mesh generation equations. For the vacuum mesh, the left and right boundaries are periodic, the bottom boundary is the plasma boundary, and the top boundary is the convex hull of the tokamak boundary. The plasma mesh uses $\alpha = 1$ and $\beta = 1$, while the vacuum mesh uses $\alpha = 120$ and $\beta = 1$.

3.2 Discretization of the Grad-Shafranov Equation

MOLE provides curvilinear divergence and gradient mimetic difference operators [2], denoted by a tilde below, which are used to solve partial differential equations on non-uniform meshes. These operators are mesh specific, so different operators must be used for the plasma domain and the vacuum domain. Thus the Grad-Shafranov equation can be discretized as

$$\left(\tilde{D}\tilde{G} - \frac{1}{r}\tilde{D}I_F^C\right)\psi_P = \begin{cases} -\mu_0 r^2 \frac{dp}{d\psi} - F\frac{dF}{d\psi} & \text{in the plasma} \\ 0 & \text{in the vacuum} \end{cases}$$

where I_F^C is the interpolation operator from the faces to the centers.

3.3 Boundary Conditions

From EFIT the value of $\psi = \psi_P + \psi_C$ is known along the plasma boundary, allowing Dirichlet boundary conditions to be used

$$\psi_P = \psi_B - \psi_C$$

where ψ_B is the value of ψ on the plasma boundary. Dirichlet boundary conditions are again used on the tokamak boundary with $\psi_P = 0$.

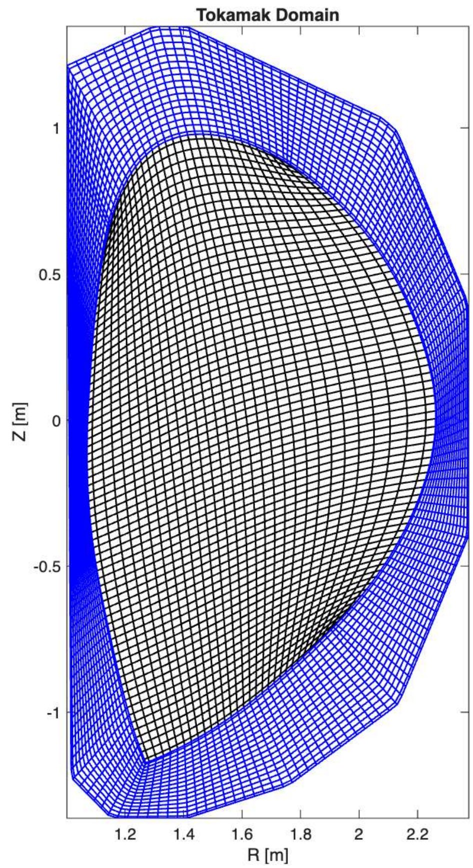

Figure 2: Cell Centers of plasma and vacuum domain

4 Results

When both solutions are interpolated onto a common 500 x 500 mesh, the MOLE solution achieves an L^2 norm of 10.304, a relative L^2 norm of 0.21563, an L^∞ norm of 0.07844, and a relative L^∞ norm of 0.56404.

Possible sources of error include the boundary conditions on the tokamak boundary and the source term for the plasma domain. Setting Dirichlet boundary conditions of 0 along the tokamak boundary implies that the plasma is sufficiently far away such that the magnetic field it produces is negligible when computing the magnetic flux there and even though the poloidal field coils have hundreds of thousands of amperes running through them, that assumption is not very accurate. Because EFIT provides $\frac{dp}{d\psi}$ and $F\frac{dF}{d\psi}$ as radial vectors, they have to be interpolated onto the plasma mesh that MOLE uses, introducing possible numerical error.

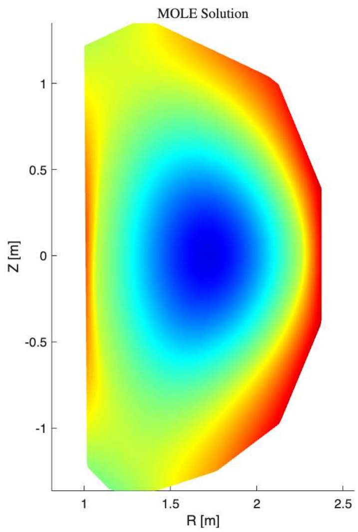

Figure 3: MOLE Solution on the tokamak domain

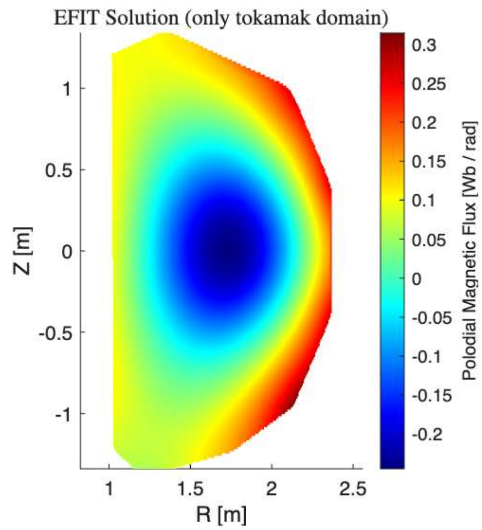

Figure 4: EFIT Solution on the tokamak domain

5 Future Work

Currently this work relies almost entirely on EFIT's output. In a realistic use case, the data ($\frac{dp}{d\psi}$, $F\frac{dF}{d\psi}$), and the plasma boundary information) are unknown, so this work needs to be adapted to compute them. This would eliminate the benefit of using two meshes (one for the plasma and one for the vacuum) as the plasma boundary could change between each nonlinear iteration and thus require a single mesh for the entire tokamak domain. Additionally there is potential expansion for optimizing the use of reference grids in generating a mesh to stay completely within the tokamak boundaries instead of the convex hull.

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