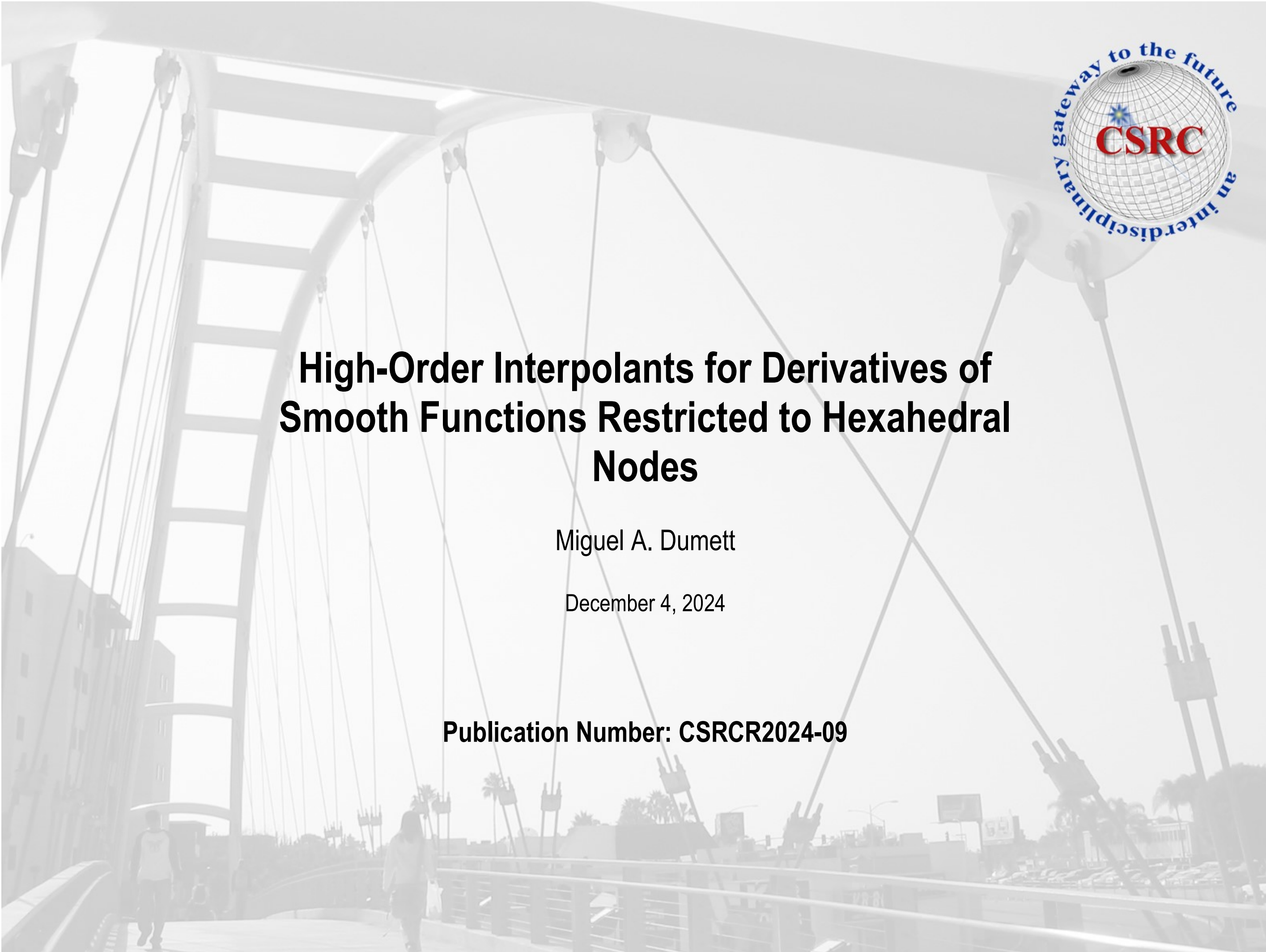

High-Order Interpolants for Derivatives of Smooth Functions Restricted to Hexahedral Nodes

Miguel A. Dumett

December 4, 2024

Publication Number: CSRCR2024-09

Computational Science & Engineering Faculty and Students Research Articles

Database Powered by the Computational Science Research Center Computing Group & Visualization Lab

COMPUTATIONAL SCIENCE & ENGINEERING

SAN DIEGO STATE UNIVERSITY

Computational Science Research Center
College of Sciences
5500 Campanile Drive
San Diego, CA 92182-1245
(619) 594-3430

High-Order Interpolants for Derivatives of Smooth Functions Restricted to Hexahedral Nodes *

Miguel A. Dumett †

December 4, 2024

Abstract

Motivated by the possibility of extending to mimetic differences several current numerical techniques to improve and/or estimate errors when the exact solution of a problem is not known, interpolation and derivative operators are introduced that approximate with uniform (including points near and at the boundary) high-order accuracy the discrete projection of a smooth function on any point of the hexahedral Cartesian domain that does not belong to the staggered grid utilized by mimetic differences. In the case of a grid point on the staggered grid, refer to [3]. They provide a generalization of the gradient, divergence, and Laplace mimetic difference operators for points not on the staggered grid. The actual stencils of these new interpolation operators are different from those of the mimetic interpolation operators. The derivations of these new interpolation operators are independent of considerations of mimetic difference operators, such as divergence, gradient, and interpolations.

1 Introduction

Mimetic differences use staggered grids for numerically approximating classic solutions of partial differential equations (PDEs). One of these two dual grids contains cell centers and some boundary points. The other one contains, depending on the dimensionality of the domain, cell vertices, edge centers, face centers. Mimetic difference gradient and divergence operators use input data from one of these two meshes and output partial derivative values on the other grid. Two mimetic interpolation operators move data from one of the meshes to the other, with uniform high-order accuracy (not only at interior grid points but also at boundary points and near them). Mimetic difference operators in two-dimensions (2D), and three-dimensions (3D) are constructed via Kronecker versions of their one-dimensional (1D) pairs.

*This work was partially supported by SDSU

†Computational Science Research Center at the San Diego State University (mdumett@sdsu.edu).

Some numerical techniques, originally developed for finite element methods, such as post-processing for elliptic equations [7], and for finite volume discretizations of elliptic, parabolic, and hyperbolic PDEs, such as adaptive mesh refinement (AMR) [2], can be extended for mimetic differences in hexahedral meshes, but these extensions require to have values of the scalar and vector fields not only on the mimetic difference dual grid points but also on several points on the boundary of each cell (e.g., post-processing) as well as on the cell interior (e.g., AMR).

This is the main purpose of this work, to provide high-order accurate interpolation and derivative formulas for arbitrary cell boundary and cell interior points in 1D domains which can be easily extended to 2D points on cell edges and interiors, as well as 3D points on cell edges, faces and interiors.

It is expected that these formulas may also play a role while trying to embed mimetic differences into a discrete exterior calculus framework [5, 4].

In the next section, 1D k^{th}-order approximation stencil weight formulas for interpolation and l-order derivatives, with $l \leq k$, and $k = 2, 4, 6, 8$, for arbitrary 1D points in $[0, 1]$ are constructed. The method can be easily extended to intervals $[a, b]$.

2 1D General Interpolation and Derivative Operators

Consider a smooth function f defined in the interval $[0, 1]$. Suppose a discrete mesh on $[0, 1]$, where the values of f are known, are given by N equal sized cells, with $h = 1/N$. The uniform nodal grid is

$$X_N = \{x_j = jh,\, j = 0, 1, \cdots, N\}.$$

With the goal of constructing high-order interpolation operators that approximate f at an arbitrary point $\alpha \in (0, 1) \setminus X_N$. If one defines the set of centers by

$$X_C = \{x_{j+1/2} = (j + 1/2)h,\, j = 0, 1, \cdots, N - 1\},$$

in [3] one can find interpolation operators of order $k = 2, 4, 6, 8$, which have a uniform degree of accuracy throughout the grid, including at points at and near the boundary, which approximate $f(\alpha)$, $\alpha \in X_C$.

Since center cells X_C are already considered in the case of staggered grids, therefore, it is enough to consider two cases:

1. $\alpha \in (x_j, x_{j+1/2})$, for some $j = 0, \cdots, N$. In this case, $\alpha = h(j_0 + a)$, for some $0 < a < \frac{1}{2}$, and $j_0 \in \{0, 1, \cdots, N - 1\}$. The stencil to be considered is

$$S(\alpha) = \{x_{j-k/2-1}, \cdots, x_{j-1}, x_j, x_{j+1}, \cdots, x_{j+k/2}\}$$

2. $\alpha \in (x_{j+1/2}, x_{j+1})$, for some $j = 0, \cdots, N$. In this case, $\alpha = h(j_0 + a)$, for some $\frac{1}{2} < a < 1$, and $j_0 \in \{0, 1, \cdots, N-1\}$. The stencil to be considered is

$$S(\alpha) = \{x_{j-k/2}, \cdots, x_{j-1}, x_j, x_{j+1}, \cdots, x_{j+k/2+1}\}$$

As pointed out, in both situations, the interpolation operator of uniform order k will use a stencil of $k+1$ points. Special treatments have to be considered when α is near the vertices 0 and 1 of the interval, because in those situations some points of the proposed stencil $S(\alpha)$ are not on the computational grid.

If α is close to the vertex 0 and $S(\alpha)$ does not include the $r < k/2+1$ first stencil points then in $S(\alpha)$, one should remove the first r nodes, and add r nodes to the right of $x_{j+k/2}$.

If α is close to the vertex 1 and $S(\alpha)$ does not include the $r < k/2+1$ last stencil point then in $S(\alpha)$, one should remove the last r nodes, and add r nodes to the left of $x_{j-k/2}$.

2.1 Second-order interpolation and derivative operators

The analysis is split into six cases.

1. $\alpha \in (x_j, x_{j+1/2})$, for some $j = 1, \cdots, N-1$.

   In this case, approximate $f^{(l)}(\alpha)$ (where (l) stands for the l-derivative, $l = 0, 1, 2$ interpreting $l = 0$ as the (0)-derivative or actual function) by a linear combination with weight w's

$$f^{(l)}(\alpha) \approx w_{j-1}^{(l)}\, f(x_{j-1}) + w_j^{(l)}\, f(x_j) + w_{j+1}^{(l)}\, f(x_{j+1}).$$

   By Taylor expansions one gets (using the fact that $\alpha = h\,(j+a)$)

$$\begin{aligned} f(x_{j-1}) &\approx f(\alpha) + (-h - a\,h)f^{'}(\alpha) + \frac{1}{2}(-h - a\,h)^2 f^{"}(\alpha) \\ f(x_j) &\approx f(\alpha) + (-a\,h)f^{'}(\alpha) + \frac{1}{2}(-a\,h)^2 f^{"}(\alpha) \\ f(x_{j+1}) &\approx f(\alpha) + (h - a\,h)f^{'}(\alpha) + \frac{1}{2}(h - a\,h)^2 f^{"}(\alpha) \end{aligned}$$

   Substituting in the linear combination, one gets

$$\begin{aligned} f(\alpha)^{(l)} &\approx [w_{j-1}^{(l)} + w_j^{(l)} + w_{j+1}^{(l)}]f(\alpha) + h[(-1-a)w_{j-1}^{(l)} - a w_j^{(l)} + (1-a)w_{j+1}^{(l)}]f^{'}(\alpha) \\ &+ \frac{h^2}{2}[(-1-a)^2 w_{j-1}^{(l)} + (-a)^2 w_j^{(l)} + (1-a)^2 w_{j+1}^{(l)}]f^{"}(\alpha). \end{aligned}$$

Matching coefficients of the respective derivatives,

$$\begin{aligned} w_{j-1}^{(l)} + w_j^{(l)} + w_{j+1}^{(l)} &= \delta_{0l} \\ (-1-a)w_{j-1}^{(l)} - a w_j^{(l)} + (1-a)w_{j+1}^{(l)} &= \frac{\delta_{1l}}{h} \\ (-1-a)^2 w_{j-1}^{(l)} + (-a)^2 w_j^{(l)} + (1-a)^2 w_{j+1}^{(l)} &= \frac{2\delta_{2l}}{h^2}, \end{aligned}$$

where $\delta_{dl} = 1$ if $d = l$, and 0 otherwise.

As a linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ -1-a & -a & 1-a \\ (-1-a)^2 & (-a)^2 & (1-a)^2 \end{bmatrix} \begin{bmatrix} w_{j-1}^{(l)} \\ w_j^{(l)} \\ w_{j+1}^{(l)} \end{bmatrix} = \begin{bmatrix} \delta_{0l} \\ \frac{\delta_{1l}}{h} \\ \frac{2\delta_{2l}}{h^2} \end{bmatrix}.$$

Since $\alpha = h\,(j+a)$, with $a < \frac{1}{2}$, one defines

$$w_<^{(l)} = (w_{j-1}^{(l)}, w_j^{(l)}, w_{j+1}^{(l)})^T,$$

and

$$e_l = \left(\delta_{0l}, \frac{\delta_{1l}}{h}, \frac{2\delta_{2l}}{h^2}\right)^T,$$

and the generator vector associated

$$g_j^< = [-1-a, \quad -a, \quad 1-a],$$

then the system becomes

$$V^T(g_j^<)\, w_<^{(l)} = e_l,$$

where $V(g_j^<)$ is the Vandermonde matrix with the generator $g_j^<$. Hence,

$$w_<^{(l)} = V^{-T}(g_j^<)\, e_l,$$

or $w_<^{(l)}$ is the scaled l-th column of $V^{-T}(g_j^<)$ by the factor $\frac{l!}{h^l}$.

Using Maple one gets

$$\begin{aligned} w_<^{(0)} &= \frac{1}{2}\,[-a(1-a), \quad 2(1-a)(1+a), \quad a(a+1)]^T \\ w_<^{(1)} &= \frac{1}{2h}[2a-1, \quad -4\,a, \quad 2a+1]^T \\ w_<^{(2)} &= \frac{1}{h^2}\,[1, \quad -2, \quad 1]^T. \end{aligned}$$

2. $\alpha \in (x_{j+1/2}, x_{j+1})$, for some $j = 1, \cdots, N-1$.

   A similar approach for $\alpha = h\,(j+a)$, $\frac{1}{2} < a < 1$, with

$$g_j^{>} = [-a, \quad 1-a, \quad 2-a],$$

   and

$$w_{>}^{(l)} = (w_j^{(l)}, w_{j+1}^{(l)}, w_{j+2}^{(l)})^T,$$

   gives

$$\begin{aligned} w_{>}^{(0)} &= \frac{1}{2}\,[(2-a)(1-a), \quad 2a(2-a), \quad -a(1-a)]^T \\ w_{>}^{(1)} &= \frac{1}{2h}\,[2a-3, \quad 4(1-a), \quad 2a-1]^T \\ w_{>}^{(2)} &= \frac{1}{h^2}\,[1, \quad -2, \quad 1]^T. \end{aligned}$$

3. $\alpha \in (x_0, x_{1/2})$.

   Follow the same formulas as in the case $\alpha \in (x_{j+1/2}, x_{j+1})$.

4. $\alpha \in (x_{1/2}, x_1)$.

   Follow the same formulas as in the case $\alpha \in (x_{j+1/2}, x_{j+1})$.

5. $\alpha \in (x_{N-1}, x_{N-1/2})$.

   Follow the same formulas as in the case $\alpha \in (x_j, x_{j+1/2})$.

6. $\alpha \in (x_{N-1/2}, x_N)$.

   Follow the same formulas as in the case $\alpha \in (x_j, x_{j+1/2})$.

These formulas are for a unique point $\alpha = h\,(j_0+a)$, for a fixed index $j_0 \in \{0, 1, \cdots, N-1\}$. In some applications, approximations of derivatives of order $l = 0, 1, 2$ of smooth functions are needed for sets of points α and not just one. That will trigger the construction of several matrices depending on the sets considered.

- In some applications, it is required to approximate the $l = 0, 1, 2$-th derivatives of a smooth function at all points (for fixed a)

$$\alpha = \{h\,(j+a),\ j = 0, 1, \cdots, N-1\}.$$

In that case, one can construct the second-order one-dimensional $l = 0, 1, 2$-th derivative matrix approximations, for $0 < a < \frac{1}{2}$, as

$$
\begin{aligned}
D_<^{0,(2)}(a) &= \frac{1}{2}\begin{bmatrix} (2-a)(1-a) & 2a(2-a) & -a(1-a) & & & \\ & -a(1-a) & 2(1-a)(1+a) & a(a+1) & & \\ & & \ddots & \ddots & \ddots & \\ & & & -a(1-a) & 2(1-a)(1+a) & a(a+1) \end{bmatrix}, \\
D_<^{1,(2)}(a) &= \frac{1}{2h}\begin{bmatrix} 2a-3 & 4(1-a) & 2a-1 & & & \\ & 2a-1 & -4a & 2a+1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 2a-1 & -4a & 2a+1 \end{bmatrix}, \\
D_<^{2,(2)}(a) &= \frac{1}{h^2}\begin{bmatrix} 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \end{bmatrix},
\end{aligned}
$$

and for $\frac{1}{2} < a < 1$, as

$$
\begin{aligned}
D_>^{0,(2)}(a) &= \frac{1}{2}\begin{bmatrix} -a(1-a) & 2(1-a)(1+a) & a(a+1) & & & \\ & \ddots & \ddots & \ddots & & \\ & & -a(1-a) & 2(1-a)(1+a) & a(a+1) & \\ & & & (2-a)(1-a) & 2a(2-a) & -a(1-a) \end{bmatrix}, \\
D_>^{1,(2)}(a) &= \frac{1}{2h}\begin{bmatrix} 2a-1 & -4a & 2a+1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & 2a-1 & -4a & 2a+1 & \\ & & & 2a-3 & 4(1-a) & 2a-1 \end{bmatrix}, \\
D_>^{2,(2)}(a) &= \frac{1}{h^2}\begin{bmatrix} 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \end{bmatrix}.
\end{aligned}
$$

In 1D error post-processing it is also necessary to consider all α points for several values of a, for example,

$$
\alpha(a_1, a_2) = \left\{ h(j+a), j = 0, 1, \cdots, N-1, a_1 = \frac{h}{3}, a_2 = \frac{2h}{3} \right\}.
$$

The matrix for $\alpha(a_1, a_2)$ interleaves rows of the matrix $\alpha(a_1)$ with rows of the matrix for $\alpha(a_2)$. In AMR one constructs the 2D and 3D interpolation and derivative matrices as Kronecker products with appropriate identity matrices.

2.2 Fourth-order interpolation and derivative operators

All cases for the location of α reduce to the following stencils.

1. Approximations using stencil

$$S(\alpha) = \{x_{j-2}, \cdots, x_{j+2}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$\begin{aligned} w_<^{(0)} \quad = \quad & \Big[\frac{a\,(a-1)\,(a-2)\,(a+1)}{24}, \\ & -\frac{a\,(a-1)\,(a-2)\,(a+2)}{6}, \\ & \frac{(a-1)\,(a-2)\,(a+2)\,(a+1)}{4}, \\ & -\frac{a\,(a-2)\,(a+2)\,(a+1)}{6}, \\ & \frac{a\,(a-1)\,(a+2)\,(a+1)}{24}\Big] \end{aligned}$$

(b) 1^{sr}-order derivative

$$\begin{aligned} w_<^{(1)} \quad = \quad & \Big[\frac{(2\,a-1)\left(a^2-a-1\right)}{12\,h}, \\ & -\frac{4\,a^3-3\,a^2-8\,a+4}{6\,h}, \\ & \frac{a\left(2\,a^2-5\right)}{2\,h}, \\ & -\frac{4\,a^3+3\,a^2-8\,a-4}{6\,h}, \\ & \frac{(2\,a+1)\left(a^2+a-1\right)}{12\,h}\Big] \end{aligned}$$

(c) 2^{nd}-order derivative

$$w_<^{(2)} = \Big[\frac{6\,a^2 - 6\,a - 1}{12\,h^2}, -\frac{6\,a^2 - 3\,a - 4}{3\,h^2}, \frac{6\,a^2 - 5}{2\,h^2}, -\frac{6\,a^2 + 3\,a - 4}{3\,h^2}, \frac{6\,a^2 + 6\,a - 1}{12\,h^2}\Big]$$

(d) 3^{rd}-order derivative

$$w_<^{(3)} = \Big[\frac{2\,a - 1}{2\,h^3}, -\frac{-1 + 4\,a}{h^3}, 6\,\frac{a}{h^3}, -\frac{1 + 4\,a}{h^3}, \frac{2\,a + 1}{2\,h^3}\Big]$$

(e) 4^{th}-order derivative

$$w_<(4) = [h^{-4}, -4\,h^{-4}, 6\,h^{-4}, -4\,h^{-4}, h^{-4}]$$

2. Approximations using stencil

$$S(\alpha) = \{x_{j-1}, \cdots, x_{j+3}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$w_<^{(0)} = \Big[\frac{a\,(a-1)\,(a-2)\,(a-3)}{24}, -\frac{(a-1)\,(a-2)\,(a-3)\,(a+1)}{6}, \frac{a\,(a-2)\,(a-3)\,(a+1)}{4}, -\frac{a\,(a-1)\,(a-3)\,(a+1)}{6}, \frac{a\,(a-1)\,(a-2)\,(a+1)}{24}\Big]$$

(b) 1^{sr}-order derivative

$$w_{<}^{(1)} = \Big[\frac{(2\,a-3)\left(a^2-3\,a+1\right)}{12\,h}, -\frac{4\,a^3-15\,a^2+10\,a+5}{6\,h}, \frac{(a-1)\left(2\,a^2-4\,a-3\right)}{2\,h}, -\frac{4\,a^3-9\,a^2-2\,a+3}{6\,h}, \frac{(2\,a-1)\left(a^2-a-1\right)}{12\,h}\Big]$$

(c) 2^{nd}-order derivative

$$w_{<}^{(2)} = \Big[\frac{6\,a^2-18\,a+11}{12\,h^2}, -\frac{6\,a^2-15\,a+5}{3\,h^2}, \frac{6\,a^2-12\,a+1}{2\,h^2}, -\frac{6\,a^2-9\,a-1}{3\,h^2}, \frac{6\,a^2-6\,a-1}{12\,h^2}\Big]$$

(d) 3^{rd}-order derivative

$$w_{<}^{(3)} = \Big[\frac{2\,a-3}{2\,h^3}, -\frac{-5+4\,a}{h^3}, 6\,\frac{a-1}{h^3}, -\frac{-3+4\,a}{h^3}, \frac{2\,a-1}{2\,h^3}\Big]$$

(e) 4^{th}-order derivative

$$w_{<}(4) = [h^{-4}, -4\,h^{-4}, 6\,h^{-4}, -4\,h^{-4}, h^{-4}]$$

3. Approximations using stencil

$$S(\alpha) = \{x_j, \cdots, x_{j+4}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$
\begin{aligned}
w_<^{(0)} = & \Big[\frac{(a-1)(a-2)(a-3)(a-4)}{24}, \\
& -\frac{a(a-2)(a-3)(a-4)}{6}, \\
& \frac{a(a-1)(a-3)(a-4)}{4}, \\
& -\frac{a(a-1)(a-2)(a-4)}{6}, \\
& \frac{a(a-1)(a-2)(a-3)}{24}\Big]
\end{aligned}
$$

(b) 1^{sr}-order derivative

$$
\begin{aligned}
w_<^{(1)} = & \Big[\frac{(2a-5)\left(a^2-5a+5\right)}{12h}, \\
& -\frac{4a^3-27a^2+52a-24}{6h}, \\
& \frac{(a-2)\left(2a^2-8a+3\right)}{2h}, \\
& -\frac{4a^3-21a^2+28a-8}{6h}, \\
& \frac{(2a-3)\left(a^2-3a+1\right)}{12h}\Big]
\end{aligned}
$$

(c) 2^{nd}-order derivative

$$
\begin{aligned}
w_<^{(2)} = & \Big[\frac{6a^2-30a+35}{12h^2}, \\
& -\frac{6a^2-27a+26}{3h^2}, \\
& \frac{6a^2-24a+19}{2h^2}, \\
& -\frac{6a^2-21a+14}{3h^2}, \\
& \frac{6a^2-18a+11}{12h^2}\Big]
\end{aligned}
$$

(d) 3^{rd}-order derivative

$$w_<^{(3)} = \Big[\frac{2\,a-5}{2\,h^3}, -\frac{-9+4\,a}{h^3}, 6\,\frac{a-2}{h^3}, -\frac{-7+4\,a}{h^3}, \frac{2\,a-3}{2\,h^3}\Big]$$

(e) 4^{th}-order derivative

$$w_<(4) = [h^{-4}, -4\,h^{-4}, 6\,h^{-4}, -4\,h^{-4}, h^{-4}]$$

4. Approximations using stencil

$$S(\alpha) = \{x_{j-3}, \cdots, x_{j+1}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$w_<^{(0)} = \Big[\frac{a\,(a-1)\,(a+2)\,(a+1)}{24}, -\frac{a\,(a-1)\,(a+3)\,(a+1)}{6}, \frac{a\,(a-1)\,(a+3)\,(a+2)}{4}, -\frac{(a-1)\,(a+3)\,(a+2)\,(a+1)}{6}, \frac{a\,(a+3)\,(a+2)\,(a+1)}{24}\Big]$$

(b) 1^{sr}-order derivative

$$w_<^{(1)} = \Big[\frac{(2\,a+1)\left(a^2+a-1\right)}{12\,h}, -\frac{4\,a^3+9\,a^2-2\,a-3}{6\,h}, \frac{(a+1)\left(2\,a^2+4\,a-3\right)}{2\,h}, -\frac{4\,a^3+15\,a^2+10\,a-5}{6\,h}, \frac{(2\,a+3)\left(a^2+3\,a+1\right)}{12\,h}\Big]$$

(c) 2^{nd}-order derivative

$$w_<^{(2)} = \Big[\frac{6\,a^2+6\,a-1}{12\,h^2}, -\frac{6\,a^2+9\,a-1}{3\,h^2}, \frac{6\,a^2+12\,a+1}{2\,h^2}, -\frac{6\,a^2+15\,a+5}{3\,h^2}, \frac{6\,a^2+18\,a+11}{12\,h^2}\Big]$$

(d) 3^{rd}-order derivative

$$w_<^{(3)} = \Big[\frac{2\,a+1}{2\,h^3}, -\frac{3+4\,a}{h^3}, 6\,\frac{a+1}{h^3}, -\frac{5+4\,a}{h^3}, \frac{2\,a+3}{2\,h^3}\Big]$$

(e) 4^{th}-order derivative

$$w_<(4) = [h^{-4}, -4\,h^{-4}, 6\,h^{-4}, -4\,h^{-4}, h^{-4}]$$

2.3 Sixth-order interpolation and derivative operators

All cases for the location of α reduce to the following stencils.

1. Approximations using stencil

$$S(\alpha) = \{x_{j-3}, \cdots, x_{j+3}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$
\begin{aligned}
w_{<}^{(0)} \quad = \quad & \Big[\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a+2)\,(a+1)}{720}, \\
& -\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a+3)\,(a+1)}{120}, \\
& \frac{a\,(a-1)\,(a-2)\,(a-3)\,(a+3)\,(a+2)}{48}, \\
& -\frac{(a-1)\,(a-2)\,(a-3)\,(a+3)\,(a+2)\,(a+1)}{36}, \\
& \frac{a\,(a-2)\,(a-3)\,(a+3)\,(a+2)\,(a+1)}{48}, \\
& -\frac{a\,(a-1)\,(a-3)\,(a+3)\,(a+2)\,(a+1)}{120}, \\
& \frac{a\,(a-1)\,(a-2)\,(a+3)\,(a+2)\,(a+1)}{720}\Big]
\end{aligned}
$$

(b) 1^{st}-order derivative

$$
\begin{aligned}
w_{<}^{(1)} \quad = \quad & \Big[\frac{(2\,a-1)\left(3\,a^4-6\,a^3-13\,a^2+16\,a+12\right)}{720\,h}, \\
& -\frac{3\,a^5-5\,a^4-20\,a^3+30\,a^2+9\,a-9}{60\,h}, \\
& \frac{6\,a^5-5\,a^4-52\,a^3+39\,a^2+72\,a-36}{48\,h}, \\
& -\frac{a\left(a^2-7\right)\left(3\,a^2-7\right)}{18\,h}, \\
& \frac{6\,a^5+5\,a^4-52\,a^3-39\,a^2+72\,a+36}{48\,h}, \\
& -\frac{3\,a^5+5\,a^4-20\,a^3-30\,a^2+9\,a+9}{60\,h}, \\
& \frac{(2\,a+1)\left(3\,a^4+6\,a^3-13\,a^2-16\,a+12\right)}{720\,h}\Big]
\end{aligned}
$$

(c) 2^{nd}-order derivative

$$w_<^{(2)} = \Big[\frac{15\,a^4 - 30\,a^3 - 30\,a^2 + 45\,a + 4}{360\,h^2}, -\frac{15\,a^4 - 20\,a^3 - 60\,a^2 + 60\,a + 9}{60\,h^2}, \frac{15\,a^4 - 10\,a^3 - 78\,a^2 + 39\,a + 36}{24\,h^2}, -\frac{15\,a^4 - 84\,a^2 + 49}{18\,h^2}, \frac{15\,a^4 + 10\,a^3 - 78\,a^2 - 39\,a + 36}{24\,h^2}, -\frac{15\,a^4 + 20\,a^3 - 60\,a^2 - 60\,a + 9}{60\,h^2}, \frac{15\,a^4 + 30\,a^3 - 30\,a^2 - 45\,a + 4}{360\,h^2}\Big]$$

(d) 3^{rd}-order derivative

$$w_<^{(3)} = \Big[\frac{(2\,a - 1)\left(2\,a^2 - 2\,a - 3\right)}{24\,h^3}, -\frac{a^3 - a^2 - 2\,a + 1}{h^3}, \frac{20\,a^3 - 10\,a^2 - 52\,a + 13}{8\,h^3}, -\frac{2\,a\left(5\,a^2 - 14\right)}{3\,h^3}, \frac{20\,a^3 + 10\,a^2 - 52\,a - 13}{8\,h^3}, -\frac{a^3 + a^2 - 2\,a - 1}{h^3}, \frac{(2\,a + 1)\left(2\,a^2 + 2\,a - 3\right)}{24\,h^3}\Big]$$

(e) 4^{th}-order derivative

$$
\begin{aligned}
w_<^{(4)} &= \Big[\frac{3\,a^2-3\,a-1}{6\,h^4},\\
&-\frac{3\,a^2-2\,a-2}{h^4},\\
&\frac{15\,a^2-5\,a-13}{2\,h^4},\\
&-\frac{30\,a^2-28}{3\,h^4},\\
&\frac{15\,a^2+5\,a-13}{2\,h^4},\\
&-\frac{3\,a^2+2\,a-2}{h^4},\\
&\frac{3\,a^2+3\,a-1}{6\,h^4}\Big]
\end{aligned}
$$

(f) 5^{th}-order derivative

$$
\begin{aligned}
w_<^{(5)} &= \Big[\frac{2\,a-1}{2\,h^5},\\
&-2\,\frac{-1+3\,a}{h^5},\\
&\frac{-5+30\,a}{2\,h^5},\\
&-20\,\frac{a}{h^5},\\
&\frac{5+30\,a}{2\,h^5},\\
&-2\,\frac{1+3\,a}{h^5},\\
&\frac{2\,a+1}{2\,h^5}\Big]
\end{aligned}
$$

(g) 6^{th}-order derivative

$$w_<^{(6)} = \left[h^{-6}, -6\,h^{-6}, 15\,h^{-6}, -20\,h^{-6}, 15\,h^{-6}, -6\,h^{-6}, h^{-6}\right]$$

2. Approximations using stencil

$$S(\alpha) = \{x_{j-2}, \cdots, x_{j+4}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$w_<^{(0)} = \Big[\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a+1)}{720},\; -\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a+2)}{120},\; \frac{(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a+2)\,(a+1)}{48},\; -\frac{a\,(a-2)\,(a-3)\,(a-4)\,(a+2)\,(a+1)}{36},\; \frac{a\,(a-1)\,(a-3)\,(a-4)\,(a+2)\,(a+1)}{48},\; -\frac{a\,(a-1)\,(a-2)\,(a-4)\,(a+2)\,(a+1)}{120},\; \frac{a\,(a-1)\,(a-2)\,(a-3)\,(a+2)\,(a+1)}{720}\Big]$$

(b) 1^{st}-order derivative

$$w_<^{(1)} = \Big[\frac{(2\,a-3)\left(3\,a^4-18\,a^3+23\,a^2+12\,a-8\right)}{720\,h},\; -\frac{3\,a^5-20\,a^4+30\,a^3+30\,a^2-76\,a+24}{60\,h},\; \frac{6\,a^5-35\,a^4+28\,a^3+105\,a^2-112\,a-28}{48\,h},\; -\frac{(a-1)\left(3\,a^2-6\,a-4\right)\left(a^2-2\,a-6\right)}{18\,h},\; \frac{6\,a^5-25\,a^4-12\,a^3+87\,a^2+4\,a-24}{48\,h},\; -\frac{3\,a^5-10\,a^4-10\,a^3+30\,a^2+4\,a-8}{60\,h},\; \frac{(2\,a-1)\left(3\,a^4-6\,a^3-13\,a^2+16\,a+12\right)}{720\,h}\Big]$$

(c) 2^{nd}-order derivative

$$w_<^{(2)} = \Big[\frac{15\,a^4-90\,a^3+150\,a^2-45\,a-26}{360\,h^2},\\ -\frac{15\,a^4-80\,a^3+90\,a^2+60\,a-76}{60\,h^2},\\ \frac{15\,a^4-70\,a^3+42\,a^2+105\,a-56}{24\,h^2},\\ -\frac{15\,a^4-60\,a^3+6\,a^2+108\,a-20}{18\,h^2},\\ \frac{15\,a^4-50\,a^3-18\,a^2+87\,a+2}{24\,h^2},\\ -\frac{15\,a^4-40\,a^3-30\,a^2+60\,a+4}{60\,h^2},\\ \frac{15\,a^4-30\,a^3-30\,a^2+45\,a+4}{360\,h^2}\Big]$$

(d) 3^{rd}-order derivative

$$w_<^{(3)} = \Big[\frac{(2\,a-3)\left(2\,a^2-6\,a+1\right)}{24\,h^3},\\ -\frac{a^3-4\,a^2+3\,a+1}{h^3},\\ \frac{20\,a^3-70\,a^2+28\,a+35}{8\,h^3},\\ -\frac{(2\,a-2)\left(5\,a^2-10\,a-9\right)}{3\,h^3},\\ \frac{20\,a^3-50\,a^2-12\,a+29}{8\,h^3},\\ -\frac{a^3-2\,a^2-a+1}{h^3},\\ \frac{(2\,a-1)\left(2\,a^2-2\,a-3\right)}{24\,h^3}\Big]$$

(e) 4^{th}-order derivative

$$w_<^{(4)} = \Big[\frac{3\,a^2-9\,a+5}{6\,h^4}, -\frac{3\,a^2-8\,a+3}{h^4}, \frac{15\,a^2-35\,a+7}{2\,h^4}, -\frac{30\,a^2-60\,a+2}{3\,h^4}, \frac{15\,a^2-25\,a-3}{2\,h^4}, -\frac{3\,a^2-4\,a-1}{h^4}, \frac{3\,a^2-3\,a-1}{6\,h^4}\Big]$$

(f) 5^{th}-order derivative

$$w_<^{(5)} = \Big[\frac{2\,a-3}{2\,h^5}, -2\,\frac{-4+3\,a}{h^5}, \frac{-35+30\,a}{2\,h^5}, -20\,\frac{a-1}{h^5}, \frac{-25+30\,a}{2\,h^5}, -2\,\frac{-2+3\,a}{h^5}, \frac{2\,a-1}{2\,h^5}\Big]$$

(g) 6^{th}-order derivative

$$w_<^{(6)} = \left[h^{-6}, -6\,h^{-6}, 15\,h^{-6}, -20\,h^{-6}, 15\,h^{-6}, -6\,h^{-6}, h^{-6}\right]$$

3. Approximation using stencil

$$S(\alpha) = \{x_j, \cdots, x_{j+6}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$w_<^{(0)} = \Big[\frac{(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a-6)}{720},\\ -\frac{a\,(a-5)\,(a-6)\,(a-2)\,(a-3)\,(a-4)}{120},\\ \frac{a\,(a-1)\,(a-3)\,(a-4)\,(a-5)\,(a-6)}{48},\\ -\frac{a\,(a-1)\,(a-2)\,(a-4)\,(a-5)\,(a-6)}{36},\\ \frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-5)\,(a-6)}{48},\\ -\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-6)}{120},\\ \frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)}{720}\Big]$$

(b) 1^{st}-order derivative

$$w_<^{(1)} = \Big[\frac{(2\,a-7)\,\big(3\,a^4-42\,a^3+203\,a^2-392\,a+252\big)}{720\,h},\\ -\frac{3\,a^5-50\,a^4+310\,a^3-870\,a^2+1044\,a-360}{60\,h},\\ \frac{6\,a^5-95\,a^4+548\,a^3-1383\,a^2+1404\,a-360}{48\,h},\\ -\frac{(a-3)\,\big(a^2-6\,a+2\big)\,\big(3\,a^2-18\,a+20\big)}{18\,h},\\ \frac{6\,a^5-85\,a^4+428\,a^3-921\,a^2+792\,a-180}{48\,h},\\ -\frac{3\,a^5-40\,a^4+190\,a^3-390\,a^2+324\,a-72}{60\,h},\\ \frac{(2\,a-5)\,\big(3\,a^4-30\,a^3+95\,a^2-100\,a+24\big)}{720\,h}\Big]$$

(c) 2^{nd}-order derivative

$$w_<^{(2)} = \Big[\frac{15\,a^4 - 210\,a^3 + 1050\,a^2 - 2205\,a + 1624}{360\,h^2},\\ -\frac{15\,a^4 - 200\,a^3 + 930\,a^2 - 1740\,a + 1044}{60\,h^2},\\ \frac{15\,a^4 - 190\,a^3 + 822\,a^2 - 1383\,a + 702}{24\,h^2},\\ -\frac{15\,a^4 - 180\,a^3 + 726\,a^2 - 1116\,a + 508}{18\,h^2},\\ \frac{15\,a^4 - 170\,a^3 + 642\,a^2 - 921\,a + 396}{24\,h^2},\\ -\frac{15\,a^4 - 160\,a^3 + 570\,a^2 - 780\,a + 324}{60\,h^2},\\ \frac{15\,a^4 - 150\,a^3 + 510\,a^2 - 675\,a + 274}{360\,h^2}\Big]$$

(d) 3^{rd}-order derivative

$$w_<^{(3)} = \Big[\frac{(2\,a-7)\left(2\,a^2 - 14\,a + 21\right)}{24\,h^3},\\ -\frac{a^3 - 10\,a^2 + 31\,a - 29}{h^3},\\ \frac{20\,a^3 - 190\,a^2 + 548\,a - 461}{8\,h^3},\\ -\frac{(2\,a-6)\left(5\,a^2 - 30\,a + 31\right)}{3\,h^3},\\ \frac{20\,a^3 - 170\,a^2 + 428\,a - 307}{8\,h^3},\\ -\frac{a^3 - 8\,a^2 + 19\,a - 13}{h^3},\\ \frac{(2\,a-5)\left(2\,a^2 - 10\,a + 9\right)}{24\,h^3}\Big]$$

(e) 4^{th}-order derivative

$$w_<^{(4)} = \Big[\frac{3\,a^2 - 21\,a + 35}{6\,h^4}, \\ -\frac{3\,a^2 - 20\,a + 31}{h^4}, \\ \frac{15\,a^2 - 95\,a + 137}{2\,h^4}, \\ -\frac{30\,a^2 - 180\,a + 242}{3\,h^4}, \\ \frac{15\,a^2 - 85\,a + 107}{2\,h^4}, \\ -\frac{3\,a^2 - 16\,a + 19}{h^4}, \\ \frac{3\,a^2 - 15\,a + 17}{6\,h^4}\Big]$$

(f) 5^{th}-order derivative

$$w_<^{(5)} = \Big[\frac{2\,a - 7}{2\,h^5}, \\ -2\,\frac{-10 + 3\,a}{h^5}, \\ \frac{-95 + 30\,a}{2\,h^5}, \\ -20\,\frac{a - 3}{h^5}, \\ \frac{-85 + 30\,a}{2\,h^5}, \\ -2\,\frac{-8 + 3\,a}{h^5}, \\ \frac{2\,a - 5}{2\,h^5}\Big]$$

(g) 6^{th}-order derivative

$$w_<^{(6)} = \left[h^{-6}, -6\,h^{-6}, 15\,h^{-6}, -20\,h^{-6}, 15\,h^{-6}, -6\,h^{-6}, h^{-6}\right]$$

4. Approximation using stencil

$$S(\alpha) = \{x_{j-1}, \cdots, x_{j+5}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$
\begin{aligned}
w_<^{(0)} \quad = \quad & \Big[\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)}{720}, \\
& -\frac{(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a+1)}{120}, \\
& \frac{a\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a+1)}{48}, \\
& -\frac{a\,(a-1)\,(a-3)\,(a-4)\,(a-5)\,(a+1)}{36}, \\
& \frac{a\,(a-1)\,(a-2)\,(a-4)\,(a-5)\,(a+1)}{48}, \\
& -\frac{a\,(a-5)\,(a-1)\,(a-2)\,(a-3)\,(a+1)}{120}, \\
& \frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a+1)}{720}\Big]
\end{aligned}
$$

(b) 1^{st}-order derivative

$$
\begin{aligned}
w_<^{(1)} \quad = \quad & \Big[\frac{(2\,a-5)\,\left(3\,a^4-30\,a^3+95\,a^2-100\,a+24\right)}{720\,h}, \\
& -\frac{3\,a^5-35\,a^4+140\,a^3-210\,a^2+49\,a+77}{60\,h}, \\
& \frac{6\,a^5-65\,a^4+228\,a^3-249\,a^2-68\,a+120}{48\,h}, \\
& -\frac{(a-2)\,\left(3\,a^2-12\,a+5\right)\,\left(a^2-4\,a-3\right)}{18\,h}, \\
& \frac{6\,a^5-55\,a^4+148\,a^3-87\,a^2-76\,a+40}{48\,h}, \\
& -\frac{3\,a^5-25\,a^4+60\,a^3-30\,a^2-31\,a+15}{60\,h}, \\
& \frac{(2\,a-3)\,\left(3\,a^4-18\,a^3+23\,a^2+12\,a-8\right)}{720\,h}\Big]
\end{aligned}
$$

(c) 2^{nd}-order derivative

$$
\begin{aligned}
w_<^{(2)} \quad = \quad & \Big[\frac{15\,a^4-150\,a^3+510\,a^2-675\,a+274}{360\,h^2}, \\
& -\frac{15\,a^4-140\,a^3+420\,a^2-420\,a+49}{60\,h^2}, \\
& \frac{15\,a^4-130\,a^3+342\,a^2-249\,a-34}{24\,h^2}, \\
& -\frac{15\,a^4-120\,a^3+276\,a^2-144\,a-47}{18\,h^2}, \\
& \frac{15\,a^4-110\,a^3+222\,a^2-87\,a-38}{24\,h^2}, \\
& -\frac{15\,a^4-100\,a^3+180\,a^2-60\,a-31}{60\,h^2}, \\
& \frac{15\,a^4-90\,a^3+150\,a^2-45\,a-26}{360\,h^2}\Big]
\end{aligned}
$$

(d) 3^{rd}-order derivative

$$
\begin{aligned}
w_<^{(3)} \quad = \quad & \Big[\frac{(2\,a-5)\left(2\,a^2-10\,a+9\right)}{24\,h^3}, \\
& -\frac{a^3-7\,a^2+14\,a-7}{h^3}, \\
& \frac{20\,a^3-130\,a^2+228\,a-83}{8\,h^3}, \\
& -\frac{(2\,a-4)\left(5\,a^2-20\,a+6\right)}{3\,h^3}, \\
& \frac{20\,a^3-110\,a^2+148\,a-29}{8\,h^3}, \\
& -\frac{a^3-5\,a^2+6\,a-1}{h^3}, \\
& \frac{(2\,a-3)\left(2\,a^2-6\,a+1\right)}{24\,h^3}\Big]
\end{aligned}
$$

(e) 4^{th}-order derivative

$$
\begin{aligned}
w_{<}^{(4)} \quad = \quad & \Big[\frac{3\,a^2-15\,a+17}{6\,h^4}, \\
& -\frac{3\,a^2-14\,a+14}{h^4}, \\
& \frac{15\,a^2-65\,a+57}{2\,h^4}, \\
& -\frac{30\,a^2-120\,a+92}{3\,h^4}, \\
& \frac{15\,a^2-55\,a+37}{2\,h^4}, \\
& -\frac{3\,a^2-10\,a+6}{h^4}, \\
& \frac{3\,a^2-9\,a+5}{6\,h^4}\Big]
\end{aligned}
$$

(f) 5^{th}-order derivative

$$
\begin{aligned}
w_{<}^{(5)} \quad = \quad & \Big[\frac{2\,a-5}{2\,h^5}, \\
& -2\,\frac{-7+3\,a}{h^5}, \\
& \frac{-65+30\,a}{2\,h^5}, \\
& -20\,\frac{a-2}{h^5}, \\
& \frac{-55+30\,a}{2\,h^5}, \\
& -2\,\frac{-5+3\,a}{h^5}, \\
& \frac{2\,a-3}{2\,h^5}\Big]
\end{aligned}
$$

(g) 6^{th}-order derivative

$$w_{<}^{(6)} = \left[h^{-6}, -6\,h^{-6}, 15\,h^{-6}, -20\,h^{-6}, 15\,h^{-6}, -6\,h^{-6}, h^{-6}\right]$$

5. Approximation using stencil

$$S(\alpha) = \{x_{j-5}, \cdots, x_{j+1}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$w_<^{(0)} = \Big[\frac{a\,(a-1)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{720},\\ -\frac{a\,(a+5)\,(a-1)\,(a+3)\,(a+2)\,(a+1)}{120},\\ \frac{a\,(a-1)\,(a+5)\,(a+4)\,(a+2)\,(a+1)}{48},\\ -\frac{a\,(a-1)\,(a+5)\,(a+4)\,(a+3)\,(a+1)}{36},\\ \frac{a\,(a-1)\,(a+5)\,(a+4)\,(a+3)\,(a+2)}{48},\\ -\frac{(a-1)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{120},\\ \frac{a\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{720}\Big]$$

(b) 1^{st}-order derivative

$$w_<^{(1)} = \Big[\frac{(2\,a+3)\,\left(3\,a^4+18\,a^3+23\,a^2-12\,a-8\right)}{720\,h},\\ -\frac{3\,a^5+25\,a^4+60\,a^3+30\,a^2-31\,a-15}{60\,h},\\ \frac{6\,a^5+55\,a^4+148\,a^3+87\,a^2-76\,a-40}{48\,h},\\ -\frac{(a+2)\,\left(3\,a^2+12\,a+5\right)\,\left(a^2+4\,a-3\right)}{18\,h},\\ \frac{6\,a^5+65\,a^4+228\,a^3+249\,a^2-68\,a-120}{48\,h},\\ -\frac{3\,a^5+35\,a^4+140\,a^3+210\,a^2+49\,a-77}{60\,h},\\ \frac{(2\,a+5)\,\left(3\,a^4+30\,a^3+95\,a^2+100\,a+24\right)}{720\,h}\Big]$$

(c) 2^{nd}-order derivative

$$w_<^{(2)} = \Big[\frac{15\,a^4+90\,a^3+150\,a^2+45\,a-26}{360\,h^2}, -\frac{15\,a^4+100\,a^3+180\,a^2+60\,a-31}{60\,h^2}, \frac{15\,a^4+110\,a^3+222\,a^2+87\,a-38}{24\,h^2}, -\frac{15\,a^4+120\,a^3+276\,a^2+144\,a-47}{18\,h^2}, \frac{15\,a^4+130\,a^3+342\,a^2+249\,a-34}{24\,h^2}, -\frac{15\,a^4+140\,a^3+420\,a^2+420\,a+49}{60\,h^2}, \frac{15\,a^4+150\,a^3+510\,a^2+675\,a+274}{360\,h^2}\Big]$$

(d) 3^{rd}-order derivative

$$w_<^{(3)} = \Big[\frac{(2\,a+3)\left(2\,a^2+6\,a+1\right)}{24\,h^3}, -\frac{a^3+5\,a^2+6\,a+1}{h^3}, \frac{20\,a^3+110\,a^2+148\,a+29}{8\,h^3}, -\frac{(2\,a+4)\left(5\,a^2+20\,a+6\right)}{3\,h^3}, \frac{20\,a^3+130\,a^2+228\,a+83}{8\,h^3}, -\frac{a^3+7\,a^2+14\,a+7}{h^3}, \frac{(2\,a+5)\left(2\,a^2+10\,a+9\right)}{24\,h^3}\Big]$$

(e) 4^{th}-order derivative

$$w_<^{(4)} = \Big[\frac{3\,a^2+9\,a+5}{6\,h^4}, -\frac{3\,a^2+10\,a+6}{h^4}, \frac{15\,a^2+55\,a+37}{2\,h^4}, -\frac{30\,a^2+120\,a+92}{3\,h^4}, \frac{15\,a^2+65\,a+57}{2\,h^4}, -\frac{3\,a^2+14\,a+14}{h^4}, \frac{3\,a^2+15\,a+17}{6\,h^4}\Big]$$

(f) 5^{th}-order derivative

$$w_<^{(5)} = \Big[\frac{2\,a+3}{2\,h^5}, -2\,\frac{5+3\,a}{h^5}, \frac{55+30\,a}{2\,h^5}, -20\,\frac{a+2}{h^5}, \frac{65+30\,a}{2\,h^5}, -2\,\frac{7+3\,a}{h^5}, \frac{2\,a+5}{2\,h^5}\Big]$$

(g) 6^{th}-order derivative

$$w_<^{(6)} = \left[h^{-6}, -6\,h^{-6}, 15\,h^{-6}, -20\,h^{-6}, 15\,h^{-6}, -6\,h^{-6}, h^{-6}\right]$$

6. Approximation using stencil

$$S(\alpha) = \{x_{j-4}, \cdots, x_{j+2}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$
\begin{aligned}
w_<^{(0)} \quad = \quad & \Big[\frac{a\,(a-1)\,(a-2)\,(a+3)\,(a+2)\,(a+1)}{720}, \\
& -\frac{a\,(a-1)\,(a-2)\,(a+4)\,(a+2)\,(a+1)}{120}, \\
& \frac{a\,(a-1)\,(a-2)\,(a+4)\,(a+3)\,(a+1)}{48}, \\
& -\frac{a\,(a-1)\,(a-2)\,(a+4)\,(a+3)\,(a+2)}{36}, \\
& \frac{(a-1)\,(a-2)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{48}, \\
& -\frac{a\,(a-2)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{120}, \\
& \frac{a\,(a-1)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{720}\Big]
\end{aligned}
$$

(b) 1^{st}-order derivative

$$
\begin{aligned}
w_<^{(1)} \quad = \quad & \Big[\frac{(2\,a+1)\,\left(3\,a^4+6\,a^3-13\,a^2-16\,a+12\right)}{720\,h}, \\
& -\frac{3\,a^5+10\,a^4-10\,a^3-30\,a^2+4\,a+8}{60\,h}, \\
& \frac{6\,a^5+25\,a^4-12\,a^3-87\,a^2+4\,a+24}{48\,h}, \\
& -\frac{(a+1)\,\left(3\,a^2+6\,a-4\right)\,\left(a^2+2\,a-6\right)}{18\,h}, \\
& \frac{6\,a^5+35\,a^4+28\,a^3-105\,a^2-112\,a+28}{48\,h}, \\
& -\frac{3\,a^5+20\,a^4+30\,a^3-30\,a^2-76\,a-24}{60\,h}, \\
& \frac{(2\,a+3)\,\left(3\,a^4+18\,a^3+23\,a^2-12\,a-8\right)}{720\,h}\Big]
\end{aligned}
$$

(c) 2^{nd}-order derivative

$$
\begin{aligned}
w_{<}^{(2)} \quad = \quad & \Big[\frac{15\,a^4+30\,a^3-30\,a^2-45\,a+4}{360\,h^2}, \\
& -\frac{15\,a^4+40\,a^3-30\,a^2-60\,a+4}{60\,h^2}, \\
& \frac{15\,a^4+50\,a^3-18\,a^2-87\,a+2}{24\,h^2}, \\
& -\frac{15\,a^4+60\,a^3+6\,a^2-108\,a-20}{18\,h^2}, \\
& \frac{15\,a^4+70\,a^3+42\,a^2-105\,a-56}{24\,h^2}, \\
& -\frac{15\,a^4+80\,a^3+90\,a^2-60\,a-76}{60\,h^2}, \\
& \frac{15\,a^4+90\,a^3+150\,a^2+45\,a-26}{360\,h^2}\Big]
\end{aligned}
$$

(d) 3^{rd}-order derivative

$$
\begin{aligned}
w_{<}^{(3)} \quad = \quad & \Big[\frac{(2\,a+1)\left(2\,a^2+2\,a-3\right)}{24\,h^3}, \\
& -\frac{a^3+2\,a^2-a-1}{h^3}, \\
& \frac{20\,a^3+50\,a^2-12\,a-29}{8\,h^3}, \\
& -\frac{(2\,a+2)\left(5\,a^2+10\,a-9\right)}{3\,h^3}, \\
& \frac{20\,a^3+70\,a^2+28\,a-35}{8\,h^3}, \\
& -\frac{a^3+4\,a^2+3\,a-1}{h^3}, \\
& \frac{(2\,a+3)\left(2\,a^2+6\,a+1\right)}{24\,h^3}\Big]
\end{aligned}
$$

(e) 4^{th}-order derivative

$$w_<^{(4)} = \Big[\frac{3\,a^2+3\,a-1}{6\,h^4}, -\frac{3\,a^2+4\,a-1}{h^4}, \frac{15\,a^2+25\,a-3}{2\,h^4}, -\frac{30\,a^2+60\,a+2}{3\,h^4}, \frac{15\,a^2+35\,a+7}{2\,h^4}, -\frac{3\,a^2+8\,a+3}{h^4}, \frac{3\,a^2+9\,a+5}{6\,h^4}\Big]$$

(f) 5^{th}-order derivative

$$w_<^{(5)} = \Big[\frac{2\,a+1}{2\,h^5}, -2\,\frac{2+3\,a}{h^5}, \frac{25+30\,a}{2\,h^5}, -20\,\frac{a+1}{h^5}, \frac{35+30\,a}{2\,h^5}, -2\,\frac{4+3\,a}{h^5}, \frac{2\,a+3}{2\,h^5}\Big]$$

(g) 6^{th}-order derivative

$$w_<^{(6)} = \left[h^{-6}, -6\,h^{-6}, 15\,h^{-6}, -20\,h^{-6}, 15\,h^{-6}, -6\,h^{-6}, h^{-6}\right]$$

2.4 Eighth-order interpolation and derivative operators

All cases for the location of α reduce to the following stencils.

1. Approximations using stencil

$$S(\alpha) = \{x_{j-4}, \cdots, x_{j+4}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$\begin{aligned} w_<^{(0)} = & \Big[\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a+3)\,(a+2)\,(a+1)}{40320}, \\ & -\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a+4)\,(a+2)\,(a+1)}{5040}, \\ & \frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a+4)\,(a+3)\,(a+1)}{1440}, \\ & -\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a+4)\,(a+3)\,(a+2)}{720}, \\ & \frac{(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{576}, \\ & -\frac{a\,(a-2)\,(a-3)\,(a-4)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{720}, \\ & \frac{a\,(a-1)\,(a-3)\,(a-4)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{1440}, \\ & -\frac{a\,(a-1)\,(a-2)\,(a-4)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{5040}, \\ & \frac{a\,(a-1)\,(a-2)\,(a-3)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{40320}\Big] \end{aligned}$$

(b) 1^{st}-order derivative

$$w_<^{(1)} = \Big[\frac{(2\,a-1)\left(a^6-3\,a^5-12\,a^4+29\,a^3+39\,a^2-54\,a-36\right)}{10080\,h},$$
$$-\frac{8\,a^7-21\,a^6-126\,a^5+315\,a^4+336\,a^3-756\,a^2-128\,a+192}{5040\,h},$$
$$\frac{4\,a^7-7\,a^6-78\,a^5+130\,a^4+338\,a^3-507\,a^2-144\,a+144}{720\,h},$$
$$-\frac{8\,a^7-7\,a^6-174\,a^5+145\,a^4+976\,a^3-732\,a^2-1152\,a+576}{720\,h},$$
$$\frac{a\left(2\,a^6-45\,a^4+273\,a^2-410\right)}{144\,h},$$
$$-\frac{8\,a^7+7\,a^6-174\,a^5-145\,a^4+976\,a^3+732\,a^2-1152\,a-576}{720\,h},$$
$$\frac{4\,a^7+7\,a^6-78\,a^5-130\,a^4+338\,a^3+507\,a^2-144\,a-144}{720\,h},$$
$$-\frac{8\,a^7+21\,a^6-126\,a^5-315\,a^4+336\,a^3+756\,a^2-128\,a-192}{5040\,h},$$
$$\frac{(2\,a+1)\left(a^6+3\,a^5-12\,a^4-29\,a^3+39\,a^2+54\,a-36\right)}{10080\,h}\Big]$$

(c) 2^{nd}-order derivative

$$w_<^{(2)} = \Big[\frac{14\,a^6-42\,a^5-105\,a^4+280\,a^3+147\,a^2-294\,a-18}{10080\,h^2},$$
$$-\frac{28\,a^6-63\,a^5-315\,a^4+630\,a^3+504\,a^2-756\,a-64}{2520\,h^2},$$
$$\frac{14\,a^6-21\,a^5-195\,a^4+260\,a^3+507\,a^2-507\,a-72}{360\,h^2},$$
$$-\frac{28\,a^6-21\,a^5-435\,a^4+290\,a^3+1464\,a^2-732\,a-576}{360\,h^2},$$
$$\frac{14\,a^6-225\,a^4+819\,a^2-410}{144\,h^2},$$
$$-\frac{28\,a^6+21\,a^5-435\,a^4-290\,a^3+1464\,a^2+732\,a-576}{360\,h^2},$$
$$\frac{14\,a^6+21\,a^5-195\,a^4-260\,a^3+507\,a^2+507\,a-72}{360\,h^2},$$
$$-\frac{28\,a^6+63\,a^5-315\,a^4-630\,a^3+504\,a^2+756\,a-64}{2520\,h^2},$$
$$\frac{14\,a^6+42\,a^5-105\,a^4-280\,a^3+147\,a^2+294\,a-18}{10080\,h^2}\Big]$$

(d) 3^{rd}-order derivative

$$w_<^{(3)} = \Big[\frac{(2\,a-1)\left(a^4-2\,a^3-6\,a^2+7\,a+7\right)}{240\,h^3},\\ -\frac{8\,a^5-15\,a^4-60\,a^3+90\,a^2+48\,a-36}{120\,h^3},\\ \frac{28\,a^5-35\,a^4-260\,a^3+260\,a^2+338\,a-169}{120\,h^3},\\ -\frac{56\,a^5-35\,a^4-580\,a^3+290\,a^2+976\,a-244}{120\,h^3},\\ \frac{a\left(14\,a^4-150\,a^2+273\right)}{24\,h^3},\\ -\frac{56\,a^5+35\,a^4-580\,a^3-290\,a^2+976\,a+244}{120\,h^3},\\ \frac{28\,a^5+35\,a^4-260\,a^3-260\,a^2+338\,a+169}{120\,h^3},\\ -\frac{8\,a^5+15\,a^4-60\,a^3-90\,a^2+48\,a+36}{120\,h^3},\\ \frac{(2\,a+1)\left(a^4+2\,a^3-6\,a^2-7\,a+7\right)}{240\,h^3}\Big]$$

(e) 4^{th}-order derivative

$$w_<^{(4)} = \Big[\frac{10\,a^4-20\,a^3-30\,a^2+40\,a+7}{240\,h^4},\\ -\frac{10\,a^4-15\,a^3-45\,a^2+45\,a+12}{30\,h^4},\\ \frac{70\,a^4-70\,a^3-390\,a^2+260\,a+169}{60\,h^4},\\ -\frac{70\,a^4-35\,a^3-435\,a^2+145\,a+244}{30\,h^4},\\ \frac{70\,a^4-450\,a^2+273}{24\,h^4},\\ -\frac{70\,a^4+35\,a^3-435\,a^2-145\,a+244}{30\,h^4},\\ \frac{70\,a^4+70\,a^3-390\,a^2-260\,a+169}{60\,h^4},\\ -\frac{10\,a^4+15\,a^3-45\,a^2-45\,a+12}{30\,h^4},\\ \frac{10\,a^4+20\,a^3-30\,a^2-40\,a+7}{240\,h^4}\Big]$$

(f) 5^{th}-order derivative

$$w_<^{(5)} = \Big[\frac{(a-2)\,(2\,a-1)\,(a+1)}{12\,h^5}, -\frac{8\,a^3-9\,a^2-18\,a+9}{6\,h^5}, \frac{28\,a^3-21\,a^2-78\,a+26}{6\,h^5}, -\frac{56\,a^3-21\,a^2-174\,a+29}{6\,h^5}, \frac{5\,a\left(14\,a^2-45\right)}{6\,h^5}, -\frac{56\,a^3+21\,a^2-174\,a-29}{6\,h^5}, \frac{28\,a^3+21\,a^2-78\,a-26}{6\,h^5}, -\frac{8\,a^3+9\,a^2-18\,a-9}{6\,h^5}, \frac{(a-1)\,(2\,a+1)\,(a+2)}{12\,h^5}\Big]$$

(g) 6^{th}-order derivative

$$w_<^{(6)} = \Big[\frac{2\,a^2-2\,a-1}{4\,h^6}, -\frac{4\,a^2-3\,a-3}{h^6}, \frac{14\,a^2-7\,a-13}{h^6}, -\frac{28\,a^2-7\,a-29}{h^6}, \frac{70\,a^2-75}{2\,h^6}, -\frac{28\,a^2+7\,a-29}{h^6}, \frac{14\,a^2+7\,a-13}{h^6}, -\frac{4\,a^2+3\,a-3}{h^6}, \frac{2\,a^2+2\,a-1}{4\,h^6}\Big]$$

(h) 7^{th}-order derivative

$$w_<^{(7)} = \Big[\frac{2\,a-1}{2\,h^7}, -\frac{-3+8\,a}{h^7}, 7\,\frac{4\,a-1}{h^7}, -7\,\frac{-1+8\,a}{h^7}, 70\,\frac{a}{h^7}, -7\,\frac{1+8\,a}{h^7}, 7\,\frac{1+4\,a}{h^7}, -\frac{3+8\,a}{h^7}, \frac{2\,a+1}{2\,h^7}\Big]$$

(i) 8^{th}-order derivative

$$w_<^{(8)} = [h^{-8}, -8\,h^{-8}, 28\,h^{-8}, -56\,h^{-8}, 70\,h^{-8}, -56\,h^{-8}, 28\,h^{-8}, -8\,h^{-8}, h^{-8}]$$

2. Approximations using stencil

$$S(\alpha) = \{x_{j-3}, \cdots, x_{j+5}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$
\begin{aligned}
w_<^{(0)} \quad = \quad & \Big[\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a+2)\,(a+1)}{40320},\\
& -\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a+3)\,(a+1)}{5040},\\
& \frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a+3)\,(a+2)}{1440},\\
& -\frac{(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a+3)\,(a+2)\,(a+1)}{720},\\
& \frac{a\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a+3)\,(a+2)\,(a+1)}{576},\\
& -\frac{a\,(a-1)\,(a-3)\,(a-4)\,(a-5)\,(a+3)\,(a+2)\,(a+1)}{720},\\
& \frac{a\,(a-1)\,(a-2)\,(a-4)\,(a-5)\,(a+3)\,(a+2)\,(a+1)}{1440},\\
& -\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-5)\,(a+3)\,(a+2)\,(a+1)}{5040},\\
& \frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a+3)\,(a+2)\,(a+1)}{40320}\Big]
\end{aligned}
$$

(b) 1^{st}-order derivative

$$
\begin{aligned}
w_<^{(1)} \quad = \quad & \Big[\frac{(2\,a-3)\,\left(a^6-9\,a^5+18\,a^4+27\,a^3-75\,a^2-18\,a+20\right)}{10080\,h},\\
& -\frac{8\,a^7-77\,a^6+168\,a^5+350\,a^4-1484\,a^3+903\,a^2+684\,a-360}{5040\,h},\\
& \frac{4\,a^7-35\,a^6+48\,a^5+275\,a^4-682\,a^3-150\,a^2+1044\,a-360}{720\,h},\\
& -\frac{8\,a^7-63\,a^6+36\,a^5+630\,a^4-924\,a^3-1323\,a^2+1888\,a+324}{720\,h},\\
& \frac{(a-1)\,\left(2\,a^6-12\,a^5-15\,a^4+140\,a^3+33\,a^2-378\,a-180\right)}{144\,h},\\
& -\frac{8\,a^7-49\,a^6-48\,a^5+550\,a^4-44\,a^3-1389\,a^2+36\,a+360}{720\,h},\\
& \frac{4\,a^7-21\,a^6-36\,a^5+225\,a^4+78\,a^3-486\,a^2-28\,a+120}{720\,h},\\
& -\frac{8\,a^7-35\,a^6-84\,a^5+350\,a^4+196\,a^3-735\,a^2-72\,a+180}{5040\,h},\\
& \frac{(2\,a-1)\,\left(a^6-3\,a^5-12\,a^4+29\,a^3+39\,a^2-54\,a-36\right)}{10080\,h}\Big]
\end{aligned}
$$

(c) 2^{nd}-order derivative

$$
\begin{aligned}
w_<^{(2)} \quad = \quad & \Big[\frac{14\,a^6 - 126\,a^5 + 315\,a^4 - 693\,a^2 + 378\,a + 94}{10080\,h^2}, \\
& -\frac{28\,a^6 - 231\,a^5 + 420\,a^4 + 700\,a^3 - 2226\,a^2 + 903\,a + 342}{2520\,h^2}, \\
& \frac{14\,a^6 - 105\,a^5 + 120\,a^4 + 550\,a^3 - 1023\,a^2 - 150\,a + 522}{360\,h^2}, \\
& -\frac{28\,a^6 - 189\,a^5 + 90\,a^4 + 1260\,a^3 - 1386\,a^2 - 1323\,a + 944}{360\,h^2}, \\
& \frac{14\,a^6 - 84\,a^5 - 15\,a^4 + 620\,a^3 - 321\,a^2 - 822\,a + 198}{144\,h^2}, \\
& -\frac{28\,a^6 - 147\,a^5 - 120\,a^4 + 1100\,a^3 - 66\,a^2 - 1389\,a + 18}{360\,h^2}, \\
& \frac{14\,a^6 - 63\,a^5 - 90\,a^4 + 450\,a^3 + 117\,a^2 - 486\,a - 14}{360\,h^2}, \\
& -\frac{28\,a^6 - 105\,a^5 - 210\,a^4 + 700\,a^3 + 294\,a^2 - 735\,a - 36}{2520\,h^2}, \\
& \frac{14\,a^6 - 42\,a^5 - 105\,a^4 + 280\,a^3 + 147\,a^2 - 294\,a - 18}{10080\,h^2}\Big]
\end{aligned}
$$

(d) 3^{rd}-order derivative

$$
\begin{aligned}
w_<^{(3)} \quad = \quad & \Big[\frac{(2\,a - 3)\left(a^4 - 6\,a^3 + 6\,a^2 + 9\,a - 3\right)}{240\,h^3}, \\
& -\frac{8\,a^5 - 55\,a^4 + 80\,a^3 + 100\,a^2 - 212\,a + 43}{120\,h^3}, \\
& \frac{28\,a^5 - 175\,a^4 + 160\,a^3 + 550\,a^2 - 682\,a - 50}{120\,h^3}, \\
& -\frac{56\,a^5 - 315\,a^4 + 120\,a^3 + 1260\,a^2 - 924\,a - 441}{120\,h^3}, \\
& \frac{(a - 1)\left(14\,a^4 - 56\,a^3 - 66\,a^2 + 244\,a + 137\right)}{24\,h^3}, \\
& -\frac{56\,a^5 - 245\,a^4 - 160\,a^3 + 1100\,a^2 - 44\,a - 463}{120\,h^3}, \\
& \frac{28\,a^5 - 105\,a^4 - 120\,a^3 + 450\,a^2 + 78\,a - 162}{120\,h^3}, \\
& -\frac{8\,a^5 - 25\,a^4 - 40\,a^3 + 100\,a^2 + 28\,a - 35}{120\,h^3}, \\
& \frac{(2\,a - 1)\left(a^4 - 2\,a^3 - 6\,a^2 + 7\,a + 7\right)}{240\,h^3}\Big]
\end{aligned}
$$

(e) 4^{th}-order derivative

$$w_<^{(4)} = \Big[\frac{10\,a^4-60\,a^3+90\,a^2-33}{240\,h^4}, -\frac{10\,a^4-55\,a^3+60\,a^2+50\,a-53}{30\,h^4}, \frac{70\,a^4-350\,a^3+240\,a^2+550\,a-341}{60\,h^4}, -\frac{70\,a^4-315\,a^3+90\,a^2+630\,a-231}{30\,h^4}, \frac{70\,a^4-280\,a^3-30\,a^2+620\,a-107}{24\,h^4}, -\frac{70\,a^4-245\,a^3-120\,a^2+550\,a-11}{30\,h^4}, \frac{70\,a^4-210\,a^3-180\,a^2+450\,a+39}{60\,h^4}, -\frac{10\,a^4-25\,a^3-30\,a^2+50\,a+7}{30\,h^4}, \frac{10\,a^4-20\,a^3-30\,a^2+40\,a+7}{240\,h^4}\Big]$$

(f) 5^{th}-order derivative

$$w_<^{(5)} = \Big[\frac{a\,(2\,a-3)\,(a-3)}{12\,h^5}, -\frac{8\,a^3-33\,a^2+24\,a+10}{6\,h^5}, \frac{28\,a^3-105\,a^2+48\,a+55}{6\,h^5}, -\frac{56\,a^3-189\,a^2+36\,a+126}{6\,h^5}, \frac{(5\,a-5)\left(14\,a^2-28\,a-31\right)}{6\,h^5}, -\frac{56\,a^3-147\,a^2-48\,a+110}{6\,h^5}, \frac{28\,a^3-63\,a^2-36\,a+45}{6\,h^5}, -\frac{8\,a^3-15\,a^2-12\,a+10}{6\,h^5}, \frac{(a-2)\,(2\,a-1)\,(a+1)}{12\,h^5}\Big]$$

(g) 6^{th}-order derivative

$$
\begin{aligned}
w_{<}^{(6)} \quad = \quad & \Big[\frac{2\,a^2-6\,a+3}{4\,h^6}, \\
& -\frac{4\,a^2-11\,a+4}{h^6}, \\
& \frac{14\,a^2-35\,a+8}{h^6}, \\
& -\frac{28\,a^2-63\,a+6}{h^6}, \\
& \frac{70\,a^2-140\,a-5}{2\,h^6}, \\
& -\frac{28\,a^2-49\,a-8}{h^6}, \\
& \frac{14\,a^2-21\,a-6}{h^6}, \\
& -\frac{4\,a^2-5\,a-2}{h^6}, \\
& \frac{2\,a^2-2\,a-1}{4\,h^6}\Big]
\end{aligned}
$$

(h) 7^{th}-order derivative

$$
\begin{aligned}
w_{<}^{(7)} \quad = \quad & \Big[\frac{2\,a-3}{2\,h^7}, \\
& -\frac{-11+8\,a}{h^7}, \\
& 7\,\frac{4\,a-5}{h^7}, \\
& -7\,\frac{-9+8\,a}{h^7}, \\
& 70\,\frac{a-1}{h^7}, \\
& -7\,\frac{-7+8\,a}{h^7}, \\
& 7\,\frac{-3+4\,a}{h^7}, \\
& -\frac{-5+8\,a}{h^7}, \\
& \frac{2\,a-1}{2\,h^7}\Big]
\end{aligned}
$$

(i) 8^{th}-order derivative

$$w_<^{(8)} = \left[h^{-8}, -8\,h^{-8}, 28\,h^{-8}, -56\,h^{-8}, 70\,h^{-8}, -56\,h^{-8}, 28\,h^{-8}, -8\,h^{-8}, h^{-8}\right]$$

3. Approximation using stencil

$$S(\alpha) = \{x_j, \cdots, x_{j+8}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$\begin{aligned} w_<^{(0)} \quad = \quad & \Big[\frac{(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a-6)\,(a-7)\,(a-8)}{40320}, \\ & -\frac{a\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a-6)\,(a-7)\,(a-8)}{5040}, \\ & \frac{a\,(a-1)\,(a-3)\,(a-4)\,(a-5)\,(a-6)\,(a-7)\,(a-8)}{1440}, \\ & -\frac{a\,(a-1)\,(a-2)\,(a-4)\,(a-5)\,(a-6)\,(a-7)\,(a-8)}{720}, \\ & \frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-5)\,(a-6)\,(a-7)\,(a-8)}{576}, \\ & -\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-6)\,(a-7)\,(a-8)}{720}, \\ & \frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a-7)\,(a-8)}{1440}, \\ & -\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a-6)\,(a-8)}{5040}, \\ & \frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a-6)\,(a-7)}{40320}\Big] \end{aligned}$$

(b) 1^{st}-order derivative

$$w_<^{(1)} = \Big[\frac{(2a-9)\left(a^6-27a^5+288a^4-1539a^3+4299a^2-5886a+3044\right)}{10080\,h},$$
$$-\frac{8a^7-245a^6+3066a^5-20125a^4+73696a^3-146580a^2+138528a-40320}{5040\,h},$$
$$\frac{4a^7-119a^6+1434a^5-8950a^4+30578a^3-55059a^2+44712a-10080}{720\,h},$$
$$-\frac{8a^7-231a^6+2682a^5-15975a^4+51456a^3-86076a^2+64096a-13440}{720\,h},$$
$$\frac{(a-4)\left(2a^6-48a^5+435a^4-1840a^3+3633a^2-2952a+630\right)}{144\,h},$$
$$-\frac{8a^7-217a^6+2346a^5-12905a^4+38176a^3-58692a^2+40608a-8064}{720\,h},$$
$$\frac{4a^7-105a^6+1098a^5-5850a^4+16818a^3-25245a^2+17144a-3360}{720\,h},$$
$$-\frac{8a^7-203a^6+2058a^5-10675a^4+30016a^3-44268a^2+29664a-5760}{5040\,h},$$
$$\frac{(2a-7)\left(a^6-21a^5+168a^4-637a^3+1155a^2-882a+180\right)}{10080\,h}\Big]$$

(c) 2^{nd}-order derivative

$$w_<^{(2)} = \Big[\frac{14a^6-378a^5+4095a^4-22680a^3+67347a^2-100926a+59062}{10080\,h^2},$$
$$-\frac{28a^6-735a^5+7665a^4-40250a^3+110544a^2-146580a+69264}{2520\,h^2},$$
$$\frac{14a^6-357a^5+3585a^4-17900a^3+45867a^2-55059a+22356}{360\,h^2},$$
$$-\frac{28a^6-693a^5+6705a^4-31950a^3+77184a^2-86076a+32048}{360\,h^2},$$
$$\frac{14a^6-336a^5+3135a^4-14320a^3+32979a^2-34968a+12438}{144\,h^2},$$
$$-\frac{28a^6-651a^5+5865a^4-25810a^3+57264a^2-58692a+20304}{360\,h^2},$$
$$\frac{14a^6-315a^5+2745a^4-11700a^3+25227a^2-25245a+8572}{360\,h^2},$$
$$-\frac{28a^6-609a^5+5145a^4-21350a^3+45024a^2-44268a+14832}{2520\,h^2},$$
$$\frac{14a^6-294a^5+2415a^4-9800a^3+20307a^2-19698a+6534}{10080\,h^2}\Big]$$

(d) 3^{rd}-order derivative

$$w_<^{(3)} = \Big[\frac{(2\,a-9)\left(a^4-18\,a^3+114\,a^2-297\,a+267\right)}{240\,h^3},$$
$$-\frac{8\,a^5-175\,a^4+1460\,a^3-5750\,a^2+10528\,a-6980}{120\,h^3},$$
$$\frac{28\,a^5-595\,a^4+4780\,a^3-17900\,a^2+30578\,a-18353}{120\,h^3},$$
$$-\frac{56\,a^5-1155\,a^4+8940\,a^3-31950\,a^2+51456\,a-28692}{120\,h^3},$$
$$\frac{(a-4)\left(14\,a^4-224\,a^3+1194\,a^2-2384\,a+1457\right)}{24\,h^3},$$
$$-\frac{56\,a^5-1085\,a^4+7820\,a^3-25810\,a^2+38176\,a-19564}{120\,h^3},$$
$$\frac{28\,a^5-525\,a^4+3660\,a^3-11700\,a^2+16818\,a-8415}{120\,h^3},$$
$$-\frac{8\,a^5-145\,a^4+980\,a^3-3050\,a^2+4288\,a-2108}{120\,h^3},$$
$$\frac{(2\,a-7)\left(a^4-14\,a^3+66\,a^2-119\,a+67\right)}{240\,h^3}\Big]$$

(e) 4^{th}-order derivative

$$w_<^{(4)} = \Big[\frac{10\,a^4-180\,a^3+1170\,a^2-3240\,a+3207}{240\,h^4},$$
$$-\frac{10\,a^4-175\,a^3+1095\,a^2-2875\,a+2632}{30\,h^4},$$
$$\frac{70\,a^4-1190\,a^3+7170\,a^2-17900\,a+15289}{60\,h^4},$$
$$-\frac{70\,a^4-1155\,a^3+6705\,a^2-15975\,a+12864}{30\,h^4},$$
$$\frac{70\,a^4-1120\,a^3+6270\,a^2-14320\,a+10993}{24\,h^4},$$
$$-\frac{70\,a^4-1085\,a^3+5865\,a^2-12905\,a+9544}{30\,h^4},$$
$$\frac{70\,a^4-1050\,a^3+5490\,a^2-11700\,a+8409}{60\,h^4},$$
$$-\frac{10\,a^4-145\,a^3+735\,a^2-1525\,a+1072}{30\,h^4},$$
$$\frac{10\,a^4-140\,a^3+690\,a^2-1400\,a+967}{240\,h^4}\Big]$$

(f) 5^{th}-order derivative

$$
\begin{aligned}
w_{<}^{(5)} \quad = \quad & \Big[\frac{(a-6)\,(2\,a-9)\,(a-3)}{12\,h^5},\\
& -\frac{8\,a^3-105\,a^2+438\,a-575}{6\,h^5},\\
& \frac{28\,a^3-357\,a^2+1434\,a-1790}{6\,h^5},\\
& -\frac{56\,a^3-693\,a^2+2682\,a-3195}{6\,h^5},\\
& \frac{(5\,a-20)\left(14\,a^2-112\,a+179\right)}{6\,h^5},\\
& -\frac{56\,a^3-651\,a^2+2346\,a-2581}{6\,h^5},\\
& \frac{28\,a^3-315\,a^2+1098\,a-1170}{6\,h^5},\\
& -\frac{8\,a^3-87\,a^2+294\,a-305}{6\,h^5},\\
& \frac{(a-5)\,(2\,a-7)\,(a-2)}{12\,h^5}\Big]
\end{aligned}
$$

(g) 6^{th}-order derivative

$$
\begin{aligned}
w_{<}^{(6)} \quad = \quad & \Big[\frac{2\,a^2-18\,a+39}{4\,h^6},\\
& -\frac{4\,a^2-35\,a+73}{h^6},\\
& \frac{14\,a^2-119\,a+239}{h^6},\\
& -\frac{28\,a^2-231\,a+447}{h^6},\\
& \frac{70\,a^2-560\,a+1045}{2\,h^6},\\
& -\frac{28\,a^2-217\,a+391}{h^6},\\
& \frac{14\,a^2-105\,a+183}{h^6},\\
& -\frac{4\,a^2-29\,a+49}{h^6},\\
& \frac{2\,a^2-14\,a+23}{4\,h^6}\Big]
\end{aligned}
$$

(h) 7^{th}-order derivative

$$w_{<}^{(7)} = \Big[\frac{2\,a-9}{2\,h^7}, -\frac{-35+8\,a}{h^7}, 7\,\frac{-17+4\,a}{h^7}, -7\,\frac{-33+8\,a}{h^7}, 70\,\frac{a-4}{h^7}, -7\,\frac{-31+8\,a}{h^7}, 7\,\frac{-15+4\,a}{h^7}, -\frac{-29+8\,a}{h^7}, \frac{2\,a-7}{2\,h^7}\Big]$$

(i) 8^{th}-order derivative

$$w_{<}^{(8)} = \left[h^{-8}, -8\,h^{-8}, 28\,h^{-8}, -56\,h^{-8}, 70\,h^{-8}, -56\,h^{-8}, 28\,h^{-8}, -8\,h^{-8}, h^{-8}\right]$$

4. Approximation using stencil

$$S(\alpha) = \{x_{j-1}, \cdots, x_{j+7}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$
\begin{aligned}
w_<^{(0)} \quad = \quad & \Big[\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a-6)\,(a-7)}{40320}, \\
& -\frac{(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a-6)\,(a-7)\,(a+1)}{5040}, \\
& \frac{a\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a-6)\,(a-7)\,(a+1)}{1440}, \\
& -\frac{a\,(a-1)\,(a-3)\,(a-4)\,(a-5)\,(a-6)\,(a-7)\,(a+1)}{720}, \\
& \frac{a\,(a-1)\,(a-2)\,(a-4)\,(a-5)\,(a-6)\,(a-7)\,(a+1)}{576}, \\
& -\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-5)\,(a-6)\,(a-7)\,(a+1)}{720}, \\
& \frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-6)\,(a-7)\,(a+1)}{1440}, \\
& -\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a-7)\,(a+1)}{5040}, \\
& \frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a-6)\,(a+1)}{40320}\Big]
\end{aligned}
$$

(b) 1^{st}-order derivative

$$
\begin{aligned}
w_<^{(1)} \quad = \quad & \Big[\frac{(2\,a-7)\,\left(a^6-21\,a^5+168\,a^4-637\,a^3+1155\,a^2-882\,a+180\right)}{10080\,h}, \\
& -\frac{8\,a^7-189\,a^6+1764\,a^5-8190\,a^4+19236\,a^3-19089\,a^2-128\,a+8028}{5040\,h}, \\
& \frac{4\,a^7-91\,a^6+804\,a^5-3425\,a^4+6878\,a^3-4386\,a^2-2988\,a+2520}{720\,h}, \\
& -\frac{8\,a^7-175\,a^6+1464\,a^5-5750\,a^4+10036\,a^3-4035\,a^2-5508\,a+2520}{720\,h}, \\
& \frac{(a-3)\,\left(2\,a^6-36\,a^5+225\,a^4-540\,a^3+273\,a^2+306\,a-140\right)}{144\,h}, \\
& -\frac{8\,a^7-161\,a^6+1212\,a^5-4150\,a^4+5956\,a^3-1221\,a^2-3384\,a+1260}{720\,h}, \\
& \frac{4\,a^7-77\,a^6+552\,a^5-1795\,a^4+2438\,a^3-402\,a^2-1404\,a+504}{720\,h}, \\
& -\frac{8\,a^7-147\,a^6+1008\,a^5-3150\,a^4+4116\,a^3-567\,a^2-2396\,a+840}{5040\,h}, \\
& \frac{(2\,a-5)\,\left(a^6-15\,a^5+78\,a^4-155\,a^3+57\,a^2+90\,a-36\right)}{10080\,h}\Big]
\end{aligned}
$$

(c) 2^{nd}-order derivative

$$
\begin{aligned}
w_<^{(2)} \quad = \quad & \Big[\frac{14\,a^6 - 294\,a^5 + 2415\,a^4 - 9800\,a^3 + 20307\,a^2 - 19698\,a + 6534}{10080\,h^2}, \\
& -\frac{28\,a^6 - 567\,a^5 + 4410\,a^4 - 16380\,a^3 + 28854\,a^2 - 19089\,a - 64}{2520\,h^2}, \\
& \frac{14\,a^6 - 273\,a^5 + 2010\,a^4 - 6850\,a^3 + 10317\,a^2 - 4386\,a - 1494}{360\,h^2}, \\
& -\frac{28\,a^6 - 525\,a^5 + 3660\,a^4 - 11500\,a^3 + 15054\,a^2 - 4035\,a - 2754}{360\,h^2}, \\
& \frac{14\,a^6 - 252\,a^5 + 1665\,a^4 - 4860\,a^3 + 5679\,a^2 - 1026\,a - 1058}{144\,h^2}, \\
& -\frac{28\,a^6 - 483\,a^5 + 3030\,a^4 - 8300\,a^3 + 8934\,a^2 - 1221\,a - 1692}{360\,h^2}, \\
& \frac{14\,a^6 - 231\,a^5 + 1380\,a^4 - 3590\,a^3 + 3657\,a^2 - 402\,a - 702}{360\,h^2}, \\
& -\frac{28\,a^6 - 441\,a^5 + 2520\,a^4 - 6300\,a^3 + 6174\,a^2 - 567\,a - 1198}{2520\,h^2}, \\
& \frac{14\,a^6 - 210\,a^5 + 1155\,a^4 - 2800\,a^3 + 2667\,a^2 - 210\,a - 522}{10080\,h^2}\Big]
\end{aligned}
$$

(d) 3^{rd}-order derivative

$$
\begin{aligned}
w_<^{(3)} \quad = \quad & \Big[\frac{(2\,a - 7)\left(a^4 - 14\,a^3 + 66\,a^2 - 119\,a + 67\right)}{240\,h^3}, \\
& -\frac{8\,a^5 - 135\,a^4 + 840\,a^3 - 2340\,a^2 + 2748\,a - 909}{120\,h^3}, \\
& \frac{28\,a^5 - 455\,a^4 + 2680\,a^3 - 6850\,a^2 + 6878\,a - 1462}{120\,h^3}, \\
& -\frac{56\,a^5 - 875\,a^4 + 4880\,a^3 - 11500\,a^2 + 10036\,a - 1345}{120\,h^3}, \\
& \frac{(a - 3)\left(14\,a^4 - 168\,a^3 + 606\,a^2 - 612\,a + 57\right)}{24\,h^3}, \\
& -\frac{56\,a^5 - 805\,a^4 + 4040\,a^3 - 8300\,a^2 + 5956\,a - 407}{120\,h^3}, \\
& \frac{28\,a^5 - 385\,a^4 + 1840\,a^3 - 3590\,a^2 + 2438\,a - 134}{120\,h^3}, \\
& -\frac{8\,a^5 - 105\,a^4 + 480\,a^3 - 900\,a^2 + 588\,a - 27}{120\,h^3}, \\
& \frac{(2\,a - 5)\left(a^4 - 10\,a^3 + 30\,a^2 - 25\,a + 1\right)}{240\,h^3}\Big]
\end{aligned}
$$

(e) 4^{th}-order derivative

$$w_<^{(4)} = \Big[\frac{10\,a^4 - 140\,a^3 + 690\,a^2 - 1400\,a + 967}{240\,h^4}, -\frac{10\,a^4 - 135\,a^3 + 630\,a^2 - 1170\,a + 687}{30\,h^4}, \frac{70\,a^4 - 910\,a^3 + 4020\,a^2 - 6850\,a + 3439}{60\,h^4}, -\frac{70\,a^4 - 875\,a^3 + 3660\,a^2 - 5750\,a + 2509}{30\,h^4}, \frac{70\,a^4 - 840\,a^3 + 3330\,a^2 - 4860\,a + 1893}{24\,h^4}, -\frac{70\,a^4 - 805\,a^3 + 3030\,a^2 - 4150\,a + 1489}{30\,h^4}, \frac{70\,a^4 - 770\,a^3 + 2760\,a^2 - 3590\,a + 1219}{60\,h^4}, -\frac{10\,a^4 - 105\,a^3 + 360\,a^2 - 450\,a + 147}{30\,h^4}, \frac{10\,a^4 - 100\,a^3 + 330\,a^2 - 400\,a + 127}{240\,h^4}\Big]$$

(f) 5^{th}-order derivative

$$w_<^{(5)} = \Big[\frac{(a-5)\,(2\,a-7)\,(a-2)}{12\,h^5}, -\frac{8\,a^3 - 81\,a^2 + 252\,a - 234}{6\,h^5}, \frac{28\,a^3 - 273\,a^2 + 804\,a - 685}{6\,h^5}, -\frac{56\,a^3 - 525\,a^2 + 1464\,a - 1150}{6\,h^5}, \frac{(5\,a-15)\left(14\,a^2 - 84\,a + 81\right)}{6\,h^5}, -\frac{56\,a^3 - 483\,a^2 + 1212\,a - 830}{6\,h^5}, \frac{28\,a^3 - 231\,a^2 + 552\,a - 359}{6\,h^5}, -\frac{8\,a^3 - 63\,a^2 + 144\,a - 90}{6\,h^5}, \frac{(2\,a-5)\,(a-1)\,(a-4)}{12\,h^5}\Big]$$

(g) 6^{th}-order derivative

$$w_{<}^{(6)} = \Big[\frac{2\,a^2 - 14\,a + 23}{4\,h^6}, \\ -\frac{4\,a^2 - 27\,a + 42}{h^6}, \\ \frac{14\,a^2 - 91\,a + 134}{h^6}, \\ -\frac{28\,a^2 - 175\,a + 244}{h^6}, \\ \frac{70\,a^2 - 420\,a + 555}{2\,h^6}, \\ -\frac{28\,a^2 - 161\,a + 202}{h^6}, \\ \frac{14\,a^2 - 77\,a + 92}{h^6}, \\ -\frac{4\,a^2 - 21\,a + 24}{h^6}, \\ \frac{2\,a^2 - 10\,a + 11}{4\,h^6}\Big]$$

(h) 7^{th}-order derivative

$$w_{<}^{(7)} = \Big[\frac{2\,a - 7}{2\,h^7}, \\ -\frac{-27 + 8\,a}{h^7}, \\ 7\,\frac{4\,a - 13}{h^7}, \\ -7\,\frac{-25 + 8\,a}{h^7}, \\ 70\,\frac{a - 3}{h^7}, \\ -7\,\frac{-23 + 8\,a}{h^7}, \\ 7\,\frac{-11 + 4\,a}{h^7}, \\ -\frac{-21 + 8\,a}{h^7}, \\ \frac{2\,a - 5}{2\,h^7}\Big]$$

(i) 8^{th}-order derivative

$$w_<^{(8)} = \left[h^{-8}, -8\,h^{-8}, 28\,h^{-8}, -56\,h^{-8}, 70\,h^{-8}, -56\,h^{-8}, 28\,h^{-8}, -8\,h^{-8}, h^{-8}\right]$$

5. Approximation using stencil

$$S(\alpha) = \{x_{j-2}, \cdots, x_{j+6}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$\begin{aligned} w_<^{(0)} \quad = \quad & \Big[\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a-6)\,(a+1)}{40320}, \\ & -\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a-6)\,(a+2)}{5040}, \\ & \frac{(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a-6)\,(a+2)\,(a+1)}{1440}, \\ & -\frac{a\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a-6)\,(a+2)\,(a+1)}{720}, \\ & \frac{a\,(a-1)\,(a-3)\,(a-4)\,(a-5)\,(a-6)\,(a+2)\,(a+1)}{576}, \\ & -\frac{a\,(a-1)\,(a-2)\,(a-4)\,(a-5)\,(a-6)\,(a+2)\,(a+1)}{720}, \\ & \frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-5)\,(a-6)\,(a+2)\,(a+1)}{1440}, \\ & -\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-6)\,(a+2)\,(a+1)}{5040}, \\ & \frac{a\,(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)\,(a+2)\,(a+1)}{40320}\Big] \end{aligned}$$

(b) 1^{st}-order derivative

$$w_<^{(1)} = \Big[\frac{(2a-5)\left(a^6-15a^5+78a^4-155a^3+57a^2+90a-36\right)}{10080h},$$
$$-\frac{8a^7-133a^6+798a^5-1925a^4+616a^3+4452a^2-5616a+1440}{5040h},$$
$$\frac{4a^7-63a^6+342a^5-630a^4-462a^3+2457a^2-1324a-684}{720h},$$
$$-\frac{8a^7-119a^6+582a^5-775a^4-1544a^3+3756a^2-144a-1440}{720h},$$
$$\frac{(a-2)\left(2a^6-24a^5+75a^4+40a^3-327a^2-36a+90\right)}{144h},$$
$$-\frac{8a^7-105a^6+414a^5-225a^4-1464a^3+1620a^2+592a-480}{720h},$$
$$\frac{4a^7-49a^6+174a^5-50a^4-622a^3+591a^2+252a-180}{720h},$$
$$-\frac{8a^7-91a^6+294a^5-35a^4-1064a^3+924a^2+432a-288}{5040h},$$
$$\frac{(2a-3)\left(a^6-9a^5+18a^4+27a^3-75a^2-18a+20\right)}{10080h}\Big]$$

(c) 2^{nd}-order derivative

$$w_<^{(2)} = \Big[\frac{14a^6-210a^5+1155a^4-2800a^3+2667a^2-210a-522}{10080h^2},$$
$$-\frac{28a^6-399a^5+1995a^4-3850a^3+924a^2+4452a-2808}{2520h^2},$$
$$\frac{14a^6-189a^5+855a^4-1260a^3-693a^2+2457a-662}{360h^2},$$
$$-\frac{28a^6-357a^5+1455a^4-1550a^3-2316a^2+3756a-72}{360h^2},$$
$$\frac{14a^6-168a^5+615a^4-440a^3-1221a^2+1236a+162}{144h^2},$$
$$-\frac{28a^6-315a^5+1035a^4-450a^3-2196a^2+1620a+296}{360h^2},$$
$$\frac{14a^6-147a^5+435a^4-100a^3-933a^2+591a+126}{360h^2},$$
$$-\frac{28a^6-273a^5+735a^4-70a^3-1596a^2+924a+216}{2520h^2},$$
$$\frac{14a^6-126a^5+315a^4-693a^2+378a+94}{10080h^2}\Big]$$

(d) 3^{rd}-order derivative

$$
\begin{aligned}
w_<^{(3)} \quad = \quad & \Big[\frac{(2\,a-5)\left(a^4-10\,a^3+30\,a^2-25\,a+1\right)}{240\,h^3}, \\
& -\frac{8\,a^5-95\,a^4+380\,a^3-550\,a^2+88\,a+212}{120\,h^3}, \\
& \frac{28\,a^5-315\,a^4+1140\,a^3-1260\,a^2-462\,a+819}{120\,h^3}, \\
& -\frac{56\,a^5-595\,a^4+1940\,a^3-1550\,a^2-1544\,a+1252}{120\,h^3}, \\
& \frac{(a-2)\left(14\,a^4-112\,a^3+186\,a^2+152\,a-103\right)}{24\,h^3}, \\
& -\frac{56\,a^5-525\,a^4+1380\,a^3-450\,a^2-1464\,a+540}{120\,h^3}, \\
& \frac{28\,a^5-245\,a^4+580\,a^3-100\,a^2-622\,a+197}{120\,h^3}, \\
& -\frac{8\,a^5-65\,a^4+140\,a^3-10\,a^2-152\,a+44}{120\,h^3}, \\
& \frac{(2\,a-3)\left(a^4-6\,a^3+6\,a^2+9\,a-3\right)}{240\,h^3}\Big]
\end{aligned}
$$

(e) 4^{th}-order derivative

$$
\begin{aligned}
w_<^{(4)} \quad = \quad & \Big[\frac{10\,a^4-100\,a^3+330\,a^2-400\,a+127}{240\,h^4}, \\
& -\frac{10\,a^4-95\,a^3+285\,a^2-275\,a+22}{30\,h^4}, \\
& \frac{70\,a^4-630\,a^3+1710\,a^2-1260\,a-231}{60\,h^4}, \\
& -\frac{70\,a^4-595\,a^3+1455\,a^2-775\,a-386}{30\,h^4}, \\
& \frac{70\,a^4-560\,a^3+1230\,a^2-440\,a-407}{24\,h^4}, \\
& -\frac{70\,a^4-525\,a^3+1035\,a^2-225\,a-366}{30\,h^4}, \\
& \frac{70\,a^4-490\,a^3+870\,a^2-100\,a-311}{60\,h^4}, \\
& -\frac{10\,a^4-65\,a^3+105\,a^2-5\,a-38}{30\,h^4}, \\
& \frac{10\,a^4-60\,a^3+90\,a^2-33}{240\,h^4}\Big]
\end{aligned}
$$

(f) 5^{th}-order derivative

$$
\begin{aligned}
w_{<}^{(5)} \quad = \quad & \Big[\frac{(2\,a-5)\,(a-1)\,(a-4)}{12\,h^5},\\
& -\frac{8\,a^3-57\,a^2+114\,a-55}{6\,h^5},\\
& \frac{28\,a^3-189\,a^2+342\,a-126}{6\,h^5},\\
& -\frac{56\,a^3-357\,a^2+582\,a-155}{6\,h^5},\\
& \frac{(5\,a-10)\left(14\,a^2-56\,a+11\right)}{6\,h^5},\\
& -\frac{56\,a^3-315\,a^2+414\,a-45}{6\,h^5},\\
& \frac{28\,a^3-147\,a^2+174\,a-10}{6\,h^5},\\
& -\frac{8\,a^3-39\,a^2+42\,a-1}{6\,h^5},\\
& \frac{a\,(2\,a-3)\,(a-3)}{12\,h^5}\Big]
\end{aligned}
$$

(g) 6^{th}-order derivative

$$
\begin{aligned}
w_{<}^{(6)} \quad = \quad & \Big[\frac{2\,a^2-10\,a+11}{4\,h^6},\\
& -\frac{4\,a^2-19\,a+19}{h^6},\\
& \frac{14\,a^2-63\,a+57}{h^6},\\
& -\frac{28\,a^2-119\,a+97}{h^6},\\
& \frac{70\,a^2-280\,a+205}{2\,h^6},\\
& -\frac{28\,a^2-105\,a+69}{h^6},\\
& \frac{14\,a^2-49\,a+29}{h^6},\\
& -\frac{4\,a^2-13\,a+7}{h^6},\\
& \frac{2\,a^2-6\,a+3}{4\,h^6}\Big]
\end{aligned}
$$

(h) 7^{th}-order derivative

$$
\begin{aligned}
w_{<}^{(7)} \quad = \quad & \Big[\frac{2\,a-5}{2\,h^7}, \\
& -\frac{-19+8\,a}{h^7}, \\
& 7\,\frac{-9+4\,a}{h^7}, \\
& -7\,\frac{-17+8\,a}{h^7}, \\
& 70\,\frac{a-2}{h^7}, \\
& -7\,\frac{-15+8\,a}{h^7}, \\
& 7\,\frac{-7+4\,a}{h^7}, \\
& -\frac{-13+8\,a}{h^7}, \\
& \frac{2\,a-3}{2\,h^7}\Big]
\end{aligned}
$$

(i) 8^{th}-order derivative

$$w_{<}^{(8)} = \left[h^{-8}, -8\,h^{-8}, 28\,h^{-8}, -56\,h^{-8}, 70\,h^{-8}, -56\,h^{-8}, 28\,h^{-8}, -8\,h^{-8}, h^{-8}\right]$$

6. Approximation using stencil

$$S(\alpha) = \{x_{j-7}, \cdots, x_{j+1}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$
\begin{aligned}
w_<^{(0)} \quad = \quad & \Big[\frac{a\,(a-1)\,(a+6)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{40320}, \\
& -\frac{a\,(a-1)\,(a+7)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{5040}, \\
& \frac{a\,(a-1)\,(a+7)\,(a+6)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{1440}, \\
& -\frac{a\,(a-1)\,(a+7)\,(a+6)\,(a+5)\,(a+3)\,(a+2)\,(a+1)}{720}, \\
& \frac{a\,(a-1)\,(a+7)\,(a+6)\,(a+5)\,(a+4)\,(a+2)\,(a+1)}{576}, \\
& -\frac{a\,(a-1)\,(a+7)\,(a+6)\,(a+5)\,(a+4)\,(a+3)\,(a+1)}{720}, \\
& \frac{a\,(a-1)\,(a+7)\,(a+6)\,(a+5)\,(a+4)\,(a+3)\,(a+2)}{1440}, \\
& -\frac{(a-1)\,(a+7)\,(a+6)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{5040}, \\
& \frac{a\,(a+7)\,(a+6)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{40320}\Big]
\end{aligned}
$$

(b) 1^{st}-order derivative

$$
\begin{aligned}
w_<^{(1)} \quad = \quad & \Big[\frac{(2\,a+5)\left(a^6+15\,a^5+78\,a^4+155\,a^3+57\,a^2-90\,a-36\right)}{10080\,h}, \\
& -\frac{8\,a^7+147\,a^6+1008\,a^5+3150\,a^4+4116\,a^3+567\,a^2-2396\,a-840}{5040\,h}, \\
& \frac{4\,a^7+77\,a^6+552\,a^5+1795\,a^4+2438\,a^3+402\,a^2-1404\,a-504}{720\,h}, \\
& -\frac{8\,a^7+161\,a^6+1212\,a^5+4150\,a^4+5956\,a^3+1221\,a^2-3384\,a-1260}{720\,h}, \\
& \frac{(a+3)\left(2\,a^6+36\,a^5+225\,a^4+540\,a^3+273\,a^2-306\,a-140\right)}{144\,h}, \\
& -\frac{8\,a^7+175\,a^6+1464\,a^5+5750\,a^4+10036\,a^3+4035\,a^2-5508\,a-2520}{720\,h}, \\
& \frac{4\,a^7+91\,a^6+804\,a^5+3425\,a^4+6878\,a^3+4386\,a^2-2988\,a-2520}{720\,h}, \\
& -\frac{8\,a^7+189\,a^6+1764\,a^5+8190\,a^4+19236\,a^3+19089\,a^2-128\,a-8028}{5040\,h}, \\
& \frac{(2\,a+7)\left(a^6+21\,a^5+168\,a^4+637\,a^3+1155\,a^2+882\,a+180\right)}{10080\,h}\Big]
\end{aligned}
$$

(c) 2^{nd}-order derivative

$$
\begin{aligned}
w_<^{(2)} = \Big[& \frac{14\,a^6+210\,a^5+1155\,a^4+2800\,a^3+2667\,a^2+210\,a-522}{10080\,h^2}, \\
& -\frac{28\,a^6+441\,a^5+2520\,a^4+6300\,a^3+6174\,a^2+567\,a-1198}{2520\,h^2}, \\
& \frac{14\,a^6+231\,a^5+1380\,a^4+3590\,a^3+3657\,a^2+402\,a-702}{360\,h^2}, \\
& -\frac{28\,a^6+483\,a^5+3030\,a^4+8300\,a^3+8934\,a^2+1221\,a-1692}{360\,h^2}, \\
& \frac{14\,a^6+252\,a^5+1665\,a^4+4860\,a^3+5679\,a^2+1026\,a-1058}{144\,h^2}, \\
& -\frac{28\,a^6+525\,a^5+3660\,a^4+11500\,a^3+15054\,a^2+4035\,a-2754}{360\,h^2}, \\
& \frac{14\,a^6+273\,a^5+2010\,a^4+6850\,a^3+10317\,a^2+4386\,a-1494}{360\,h^2}, \\
& -\frac{28\,a^6+567\,a^5+4410\,a^4+16380\,a^3+28854\,a^2+19089\,a-64}{2520\,h^2}, \\
& \frac{14\,a^6+294\,a^5+2415\,a^4+9800\,a^3+20307\,a^2+19698\,a+6534}{10080\,h^2} \Big]
\end{aligned}
$$

(d) 3^{rd}-order derivative

$$
\begin{aligned}
w_<^{(3)} = \Big[& \frac{(2\,a+5)\left(a^4+10\,a^3+30\,a^2+25\,a+1\right)}{240\,h^3}, \\
& -\frac{8\,a^5+105\,a^4+480\,a^3+900\,a^2+588\,a+27}{120\,h^3}, \\
& \frac{28\,a^5+385\,a^4+1840\,a^3+3590\,a^2+2438\,a+134}{120\,h^3}, \\
& -\frac{56\,a^5+805\,a^4+4040\,a^3+8300\,a^2+5956\,a+407}{120\,h^3}, \\
& \frac{(a+3)\left(14\,a^4+168\,a^3+606\,a^2+612\,a+57\right)}{24\,h^3}, \\
& -\frac{56\,a^5+875\,a^4+4880\,a^3+11500\,a^2+10036\,a+1345}{120\,h^3}, \\
& \frac{28\,a^5+455\,a^4+2680\,a^3+6850\,a^2+6878\,a+1462}{120\,h^3}, \\
& -\frac{8\,a^5+135\,a^4+840\,a^3+2340\,a^2+2748\,a+909}{120\,h^3}, \\
& \frac{(2\,a+7)\left(a^4+14\,a^3+66\,a^2+119\,a+67\right)}{240\,h^3} \Big]
\end{aligned}
$$

(e) 4^{th}-order derivative

$$w_<^{(4)} = \Big[\frac{10\,a^4+100\,a^3+330\,a^2+400\,a+127}{240\,h^4}, -\frac{10\,a^4+105\,a^3+360\,a^2+450\,a+147}{30\,h^4}, \frac{70\,a^4+770\,a^3+2760\,a^2+3590\,a+1219}{60\,h^4}, -\frac{70\,a^4+805\,a^3+3030\,a^2+4150\,a+1489}{30\,h^4}, \frac{70\,a^4+840\,a^3+3330\,a^2+4860\,a+1893}{24\,h^4}, -\frac{70\,a^4+875\,a^3+3660\,a^2+5750\,a+2509}{30\,h^4}, \frac{70\,a^4+910\,a^3+4020\,a^2+6850\,a+3439}{60\,h^4}, -\frac{10\,a^4+135\,a^3+630\,a^2+1170\,a+687}{30\,h^4}, \frac{10\,a^4+140\,a^3+690\,a^2+1400\,a+967}{240\,h^4}\Big]$$

(f) 5^{th}-order derivative

$$w_<^{(5)} = \Big[\frac{(2\,a+5)\,(a+4)\,(a+1)}{12\,h^5}, -\frac{8\,a^3+63\,a^2+144\,a+90}{6\,h^5}, \frac{28\,a^3+231\,a^2+552\,a+359}{6\,h^5}, -\frac{56\,a^3+483\,a^2+1212\,a+830}{6\,h^5}, \frac{(5\,a+15)\left(14\,a^2+84\,a+81\right)}{6\,h^5}, -\frac{56\,a^3+525\,a^2+1464\,a+1150}{6\,h^5}, \frac{28\,a^3+273\,a^2+804\,a+685}{6\,h^5}, -\frac{8\,a^3+81\,a^2+252\,a+234}{6\,h^5}, \frac{(a+5)\,(a+2)\,(2\,a+7)}{12\,h^5}\Big]$$

(g) 6^{th}-order derivative

$$w_<^{(6)} = \Big[\frac{2\,a^2+10\,a+11}{4\,h^6}, -\frac{4\,a^2+21\,a+24}{h^6}, \frac{14\,a^2+77\,a+92}{h^6}, -\frac{28\,a^2+161\,a+202}{h^6}, \frac{70\,a^2+420\,a+555}{2\,h^6}, -\frac{28\,a^2+175\,a+244}{h^6}, \frac{14\,a^2+91\,a+134}{h^6}, -\frac{4\,a^2+27\,a+42}{h^6}, \frac{2\,a^2+14\,a+23}{4\,h^6}\Big]$$

(h) 7^{th}-order derivative

$$w_<^{(7)} = \Big[\frac{2\,a+5}{2\,h^7}, -\frac{21+8\,a}{h^7}, 7\,\frac{11+4\,a}{h^7}, -7\,\frac{23+8\,a}{h^7}, 70\,\frac{a+3}{h^7}, -7\,\frac{25+8\,a}{h^7}, 7\,\frac{4\,a+13}{h^7}, -\frac{27+8\,a}{h^7}, \frac{2\,a+7}{2\,h^7}\Big]$$

(i) 8^{th}-order derivative

$$w_{<}^{(8)} = \left[h^{-8}, -8\,h^{-8}, 28\,h^{-8}, -56\,h^{-8}, 70\,h^{-8}, -56\,h^{-8}, 28\,h^{-8}, -8\,h^{-8}, h^{-8}\right]$$

7. Approximation using stencil

$$S(\alpha) = \{x_{j-6}, \cdots, x_{j+2}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$\begin{aligned} w_{<}^{(0)} \quad = \quad & \Big[\frac{a\,(a-1)\,(a-2)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{40320}, \\ & -\frac{a\,(a-1)\,(a-2)\,(a+6)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{5040}, \\ & \frac{a\,(a-1)\,(a-2)\,(a+6)\,(a+5)\,(a+3)\,(a+2)\,(a+1)}{1440}, \\ & -\frac{a\,(a-1)\,(a-2)\,(a+6)\,(a+5)\,(a+4)\,(a+2)\,(a+1)}{720}, \\ & \frac{a\,(a-1)\,(a-2)\,(a+6)\,(a+5)\,(a+4)\,(a+3)\,(a+1)}{576}, \\ & -\frac{a\,(a-1)\,(a-2)\,(a+6)\,(a+5)\,(a+4)\,(a+3)\,(a+2)}{720}, \\ & \frac{(a-1)\,(a-2)\,(a+6)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{1440}, \\ & -\frac{a\,(a-2)\,(a+6)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{5040}, \\ & \frac{a\,(a-1)\,(a+6)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{40320}\Big] \end{aligned}$$

(b) 1^{st}-order derivative

$$w_<^{(1)} = \Big[\frac{(2\,a+3)\left(a^6+9\,a^5+18\,a^4-27\,a^3-75\,a^2+18\,a+20\right)}{10080\,h},$$
$$-\frac{8\,a^7+91\,a^6+294\,a^5+35\,a^4-1064\,a^3-924\,a^2+432\,a+288}{5040\,h},$$
$$\frac{4\,a^7+49\,a^6+174\,a^5+50\,a^4-622\,a^3-591\,a^2+252\,a+180}{720\,h},$$
$$-\frac{8\,a^7+105\,a^6+414\,a^5+225\,a^4-1464\,a^3-1620\,a^2+592\,a+480}{720\,h},$$
$$\frac{(a+2)\left(2\,a^6+24\,a^5+75\,a^4-40\,a^3-327\,a^2+36\,a+90\right)}{144\,h},$$
$$-\frac{8\,a^7+119\,a^6+582\,a^5+775\,a^4-1544\,a^3-3756\,a^2-144\,a+1440}{720\,h},$$
$$\frac{4\,a^7+63\,a^6+342\,a^5+630\,a^4-462\,a^3-2457\,a^2-1324\,a+684}{720\,h},$$
$$-\frac{8\,a^7+133\,a^6+798\,a^5+1925\,a^4+616\,a^3-4452\,a^2-5616\,a-1440}{5040\,h},$$
$$\frac{(2\,a+5)\left(a^6+15\,a^5+78\,a^4+155\,a^3+57\,a^2-90\,a-36\right)}{10080\,h}\Big]$$

(c) 2^{nd}-order derivative

$$w_<^{(2)} = \Big[\frac{14\,a^6+126\,a^5+315\,a^4-693\,a^2-378\,a+94}{10080\,h^2},$$
$$-\frac{28\,a^6+273\,a^5+735\,a^4+70\,a^3-1596\,a^2-924\,a+216}{2520\,h^2},$$
$$\frac{14\,a^6+147\,a^5+435\,a^4+100\,a^3-933\,a^2-591\,a+126}{360\,h^2},$$
$$-\frac{28\,a^6+315\,a^5+1035\,a^4+450\,a^3-2196\,a^2-1620\,a+296}{360\,h^2},$$
$$\frac{14\,a^6+168\,a^5+615\,a^4+440\,a^3-1221\,a^2-1236\,a+162}{144\,h^2},$$
$$-\frac{28\,a^6+357\,a^5+1455\,a^4+1550\,a^3-2316\,a^2-3756\,a-72}{360\,h^2},$$
$$\frac{14\,a^6+189\,a^5+855\,a^4+1260\,a^3-693\,a^2-2457\,a-662}{360\,h^2},$$
$$-\frac{28\,a^6+399\,a^5+1995\,a^4+3850\,a^3+924\,a^2-4452\,a-2808}{2520\,h^2},$$
$$\frac{14\,a^6+210\,a^5+1155\,a^4+2800\,a^3+2667\,a^2+210\,a-522}{10080\,h^2}\Big]$$

(d) 3^{rd}-order derivative

$$
\begin{aligned}
w_<^{(3)} \quad = \quad & \Big[\frac{(2\,a+3)\left(a^4+6\,a^3+6\,a^2-9\,a-3\right)}{240\,h^3}, \\
& -\frac{8\,a^5+65\,a^4+140\,a^3+10\,a^2-152\,a-44}{120\,h^3}, \\
& \frac{28\,a^5+245\,a^4+580\,a^3+100\,a^2-622\,a-197}{120\,h^3}, \\
& -\frac{56\,a^5+525\,a^4+1380\,a^3+450\,a^2-1464\,a-540}{120\,h^3}, \\
& \frac{(a+2)\left(14\,a^4+112\,a^3+186\,a^2-152\,a-103\right)}{24\,h^3}, \\
& -\frac{56\,a^5+595\,a^4+1940\,a^3+1550\,a^2-1544\,a-1252}{120\,h^3}, \\
& \frac{28\,a^5+315\,a^4+1140\,a^3+1260\,a^2-462\,a-819}{120\,h^3}, \\
& -\frac{8\,a^5+95\,a^4+380\,a^3+550\,a^2+88\,a-212}{120\,h^3}, \\
& \frac{(2\,a+5)\left(a^4+10\,a^3+30\,a^2+25\,a+1\right)}{240\,h^3}\Big]
\end{aligned}
$$

(e) 4^{th}-order derivative

$$
\begin{aligned}
w_<^{(4)} \quad = \quad & \Big[\frac{10\,a^4+60\,a^3+90\,a^2-33}{240\,h^4}, \\
& -\frac{10\,a^4+65\,a^3+105\,a^2+5\,a-38}{30\,h^4}, \\
& \frac{70\,a^4+490\,a^3+870\,a^2+100\,a-311}{60\,h^4}, \\
& -\frac{70\,a^4+525\,a^3+1035\,a^2+225\,a-366}{30\,h^4}, \\
& \frac{70\,a^4+560\,a^3+1230\,a^2+440\,a-407}{24\,h^4}, \\
& -\frac{70\,a^4+595\,a^3+1455\,a^2+775\,a-386}{30\,h^4}, \\
& \frac{70\,a^4+630\,a^3+1710\,a^2+1260\,a-231}{60\,h^4}, \\
& -\frac{10\,a^4+95\,a^3+285\,a^2+275\,a+22}{30\,h^4}, \\
& \frac{10\,a^4+100\,a^3+330\,a^2+400\,a+127}{240\,h^4}\Big]
\end{aligned}
$$

(f) 5^{th}-order derivative

$$w_<^{(5)} = \Big[\frac{a\,(a+3)\,(2\,a+3)}{12\,h^5} - \frac{8\,a^3+39\,a^2+42\,a+1}{6\,h^5}, \frac{28\,a^3+147\,a^2+174\,a+10}{6\,h^5}, -\frac{56\,a^3+315\,a^2+414\,a+45}{6\,h^5}, \frac{(5\,a+10)\,\big(14\,a^2+56\,a+11\big)}{6\,h^5}, -\frac{56\,a^3+357\,a^2+582\,a+155}{6\,h^5}, \frac{28\,a^3+189\,a^2+342\,a+126}{6\,h^5}, -\frac{8\,a^3+57\,a^2+114\,a+55}{6\,h^5}, \frac{(2\,a+5)\,(a+4)\,(a+1)}{12\,h^5}\Big]$$

(g) 6^{th}-order derivative

$$w_<^{(6)} = \Big[\frac{2\,a^2+6\,a+3}{4\,h^6}, -\frac{4\,a^2+13\,a+7}{h^6}, \frac{14\,a^2+49\,a+29}{h^6}, -\frac{28\,a^2+105\,a+69}{h^6}, \frac{70\,a^2+280\,a+205}{2\,h^6}, -\frac{28\,a^2+119\,a+97}{h^6}, \frac{14\,a^2+63\,a+57}{h^6}, -\frac{4\,a^2+19\,a+19}{h^6}, \frac{2\,a^2+10\,a+11}{4\,h^6}\Big]$$

(h) 7^{th}-order derivative

$$w_<^{(7)} = \Big[\frac{2\,a+3}{2\,h^7}, -\frac{13+8\,a}{h^7}, 7\,\frac{7+4\,a}{h^7}, -7\,\frac{15+8\,a}{h^7}, 70\,\frac{a+2}{h^7}, -7\,\frac{17+8\,a}{h^7}, 7\,\frac{9+4\,a}{h^7}, -\frac{19+8\,a}{h^7}, \frac{2\,a+5}{2\,h^7}\Big]$$

(i) 8^{th}-order derivative

$$w_<^{(8)} = \left[h^{-8}, -8\,h^{-8}, 28\,h^{-8}, -56\,h^{-8}, 70\,h^{-8}, -56\,h^{-8}, 28\,h^{-8}, -8\,h^{-8}, h^{-8}\right]$$

8. Approximation using stencil

$$S(\alpha) = \{x_{j-5}, \cdots, x_{j+3}\}.$$

(a) 0^{th}-order derivative (or interpolation)

$$w_<^{(0)} = \Big[\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{40320},\\
-\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a+5)\,(a+3)\,(a+2)\,(a+1)}{5040},\\
\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a+5)\,(a+4)\,(a+2)\,(a+1)}{1440},\\
-\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a+5)\,(a+4)\,(a+3)\,(a+1)}{720},\\
\frac{a\,(a-1)\,(a-2)\,(a-3)\,(a+5)\,(a+4)\,(a+3)\,(a+2)}{576},\\
-\frac{(a-1)\,(a-2)\,(a-3)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{720},\\
\frac{a\,(a-2)\,(a-3)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{1440},\\
-\frac{a\,(a-1)\,(a-3)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{5040},\\
\frac{a\,(a-1)\,(a-2)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}{40320}\Big]$$

(b) 1^{st}-order derivative

$$w_<^{(1)} = \Big[\frac{(2\,a+1)\left(a^6+3\,a^5-12\,a^4-29\,a^3+39\,a^2+54\,a-36\right)}{10080\,h},\\
-\frac{8\,a^7+35\,a^6-84\,a^5-350\,a^4+196\,a^3+735\,a^2-72\,a-180}{5040\,h},\\
\frac{4\,a^7+21\,a^6-36\,a^5-225\,a^4+78\,a^3+486\,a^2-28\,a-120}{720\,h},\\
-\frac{8\,a^7+49\,a^6-48\,a^5-550\,a^4-44\,a^3+1389\,a^2+36\,a-360}{720\,h},\\
\frac{(a+1)\left(2\,a^6+12\,a^5-15\,a^4-140\,a^3+33\,a^2+378\,a-180\right)}{144\,h},\\
-\frac{8\,a^7+63\,a^6+36\,a^5-630\,a^4-924\,a^3+1323\,a^2+1888\,a-324}{720\,h},\\
\frac{4\,a^7+35\,a^6+48\,a^5-275\,a^4-682\,a^3+150\,a^2+1044\,a+360}{720\,h},\\
-\frac{8\,a^7+77\,a^6+168\,a^5-350\,a^4-1484\,a^3-903\,a^2+684\,a+360}{5040\,h},\\
\frac{(2\,a+3)\left(a^6+9\,a^5+18\,a^4-27\,a^3-75\,a^2+18\,a+20\right)}{10080\,h}\Big]$$

(c) 2^{nd}-order derivative

$$w_<^{(2)} = \Big[\frac{14\,a^6+42\,a^5-105\,a^4-280\,a^3+147\,a^2+294\,a-18}{10080\,h^2}, \\ -\frac{28\,a^6+105\,a^5-210\,a^4-700\,a^3+294\,a^2+735\,a-36}{2520\,h^2}, \\ \frac{14\,a^6+63\,a^5-90\,a^4-450\,a^3+117\,a^2+486\,a-14}{360\,h^2}, \\ -\frac{28\,a^6+147\,a^5-120\,a^4-1100\,a^3-66\,a^2+1389\,a+18}{360\,h^2}, \\ \frac{14\,a^6+84\,a^5-15\,a^4-620\,a^3-321\,a^2+822\,a+198}{144\,h^2}, \\ -\frac{28\,a^6+189\,a^5+90\,a^4-1260\,a^3-1386\,a^2+1323\,a+944}{360\,h^2}, \\ \frac{14\,a^6+105\,a^5+120\,a^4-550\,a^3-1023\,a^2+150\,a+522}{360\,h^2}, \\ -\frac{28\,a^6+231\,a^5+420\,a^4-700\,a^3-2226\,a^2-903\,a+342}{2520\,h^2}, \\ \frac{14\,a^6+126\,a^5+315\,a^4-693\,a^2-378\,a+94}{10080\,h^2}\Big]$$

(d) 3^{rd}-order derivative

$$w_<^{(3)} = \Big[\frac{(2\,a+1)\left(a^4+2\,a^3-6\,a^2-7\,a+7\right)}{240\,h^3}, \\ -\frac{8\,a^5+25\,a^4-40\,a^3-100\,a^2+28\,a+35}{120\,h^3}, \\ \frac{28\,a^5+105\,a^4-120\,a^3-450\,a^2+78\,a+162}{120\,h^3}, \\ -\frac{56\,a^5+245\,a^4-160\,a^3-1100\,a^2-44\,a+463}{120\,h^3}, \\ \frac{(a+1)\left(14\,a^4+56\,a^3-66\,a^2-244\,a+137\right)}{24\,h^3}, \\ -\frac{56\,a^5+315\,a^4+120\,a^3-1260\,a^2-924\,a+441}{120\,h^3}, \\ \frac{28\,a^5+175\,a^4+160\,a^3-550\,a^2-682\,a+50}{120\,h^3}, \\ -\frac{8\,a^5+55\,a^4+80\,a^3-100\,a^2-212\,a-43}{120\,h^3}, \\ \frac{(2\,a+3)\left(a^4+6\,a^3+6\,a^2-9\,a-3\right)}{240\,h^3}\Big]$$

(e) 4^{th}-order derivative

$$w_<^{(4)} = \Big[\frac{10\,a^4+20\,a^3-30\,a^2-40\,a+7}{240\,h^4}, -\frac{10\,a^4+25\,a^3-30\,a^2-50\,a+7}{30\,h^4}, \frac{70\,a^4+210\,a^3-180\,a^2-450\,a+39}{60\,h^4}, -\frac{70\,a^4+245\,a^3-120\,a^2-550\,a-11}{30\,h^4}, \frac{70\,a^4+280\,a^3-30\,a^2-620\,a-107}{24\,h^4}, -\frac{70\,a^4+315\,a^3+90\,a^2-630\,a-231}{30\,h^4}, \frac{70\,a^4+350\,a^3+240\,a^2-550\,a-341}{60\,h^4}, -\frac{10\,a^4+55\,a^3+60\,a^2-50\,a-53}{30\,h^4}, \frac{10\,a^4+60\,a^3+90\,a^2-33}{240\,h^4}\Big]$$

(f) 5^{th}-order derivative

$$w_<^{(5)} = \Big[\frac{(a-1)\,(2\,a+1)\,(a+2)}{12\,h^5}, -\frac{8\,a^3+15\,a^2-12\,a-10}{6\,h^5}, \frac{28\,a^3+63\,a^2-36\,a-45}{6\,h^5}, -\frac{56\,a^3+147\,a^2-48\,a-110}{6\,h^5}, \frac{(5\,a+5)\left(14\,a^2+28\,a-31\right)}{6\,h^5}, -\frac{56\,a^3+189\,a^2+36\,a-126}{6\,h^5}, \frac{28\,a^3+105\,a^2+48\,a-55}{6\,h^5}, -\frac{8\,a^3+33\,a^2+24\,a-10}{6\,h^5}, \frac{a\,(a+3)\,(2\,a+3)}{12\,h^5}\Big]$$

(g) 6^{th}-order derivative

$$w_{<}^{(6)} = \Big[\frac{2\,a^2+2\,a-1}{4\,h^6}, -\frac{4\,a^2+5\,a-2}{h^6}, \frac{14\,a^2+21\,a-6}{h^6}, -\frac{28\,a^2+49\,a-8}{h^6}, \frac{70\,a^2+140\,a-5}{2\,h^6}, -\frac{28\,a^2+63\,a+6}{h^6}, \frac{14\,a^2+35\,a+8}{h^6}, -\frac{4\,a^2+11\,a+4}{h^6}, \frac{2\,a^2+6\,a+3}{4\,h^6}\Big]$$

(h) 7^{th}-order derivative

$$w_{<}^{(7)} = \Big[\frac{2\,a+1}{2\,h^7}, -\frac{5+8\,a}{h^7}, 7\,\frac{3+4\,a}{h^7}, -7\,\frac{7+8\,a}{h^7}, 70\,\frac{a+1}{h^7}, -7\,\frac{9+8\,a}{h^7}, 7\,\frac{5+4\,a}{h^7}, -\frac{11+8\,a}{h^7}, \frac{2\,a+3}{2\,h^7}\Big]$$

(i) 8^{th}-order derivative

$$w_{<}^{(8)} = \left[h^{-8}, -8\,h^{-8}, 28\,h^{-8}, -56\,h^{-8}, 70\,h^{-8}, -56\,h^{-8}, 28\,h^{-8}, -8\,h^{-8}, h^{-8}\right]$$

References

[1] Belousov, D., and Lisitsa, V., Mimetic finite differences for boundaries misaligned with grid nodes, Journal of Computational and Applied Mathematics, Volume 428, 15 August 2023, 115185, https://doi.org/10.1016/j.cam.2023.115185.

[2] Colella, P., High-order finite-volume methods on locally-structured grids, Discrete and Continuous Dynamical Systems, 36(8),4247-4270, August 2016, doi:10.3934/dcds.2016.36.4247.

[3] Dumett, M., and Castillo, J.E., Interpolation on Staggered Grids, CSRC Report, 6-Dec-2022.

[4] Hirani, A. H., Discrete Exterior Calculus, California Institute of Technology, PhD thesis, 2003.

[5] Kreeft, J., Palha, A., and Gerritsma, M., Mimetic framework on curvilinear quadrilaterals of arbitrary order, arXiv:1111.4304 [math.NA], https://doi.org/10.48550/arXiv.1111.4304.

[6] Rawashdeh E.A., A simple method for finding the inverse of a Vandermonde matrix, Matematički Vesnik, September 2019.

[7] Varela, J., Ahmed, E., Keilegavlen, E., Nordbotten J. M., and Radu, F. A., A posteriori error estimates for hierarchical mixed-dimensional elliptic equations, J. Numer. Math. 2023; 31(4): 247-280, https://doi.org/10.1515/jnma-2022-0038.