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SIR Model For Rabies Among Foxes Using Mimetic Differences

Thomas Stone *and Miguel A. Dumett †‡

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Abstract

In this paper, mimetic operators from the Matlab MOLE library [2] are utilized to numerically solve a PDE for the SIR model. SIR models are typically used to model the spread of a disease amongst a population.

1 Introduction

In 1939 there was an epidemic of rabies amongst foxes starting in Poland which was sweeping across Europe. The wavefront of the disease advanced by several dozen kilometers each year. By 1967 the fox rabies epidemic had reached Switzerland traveling over 973 kilometers with no signs of stopping. During this time there was a similar rabies epidemic spreading across America by raccoons.

Rabies is a virus that will spread from the infection site, through biting or scratching, to the central nervous system. Rabies can take anywhere from a week to several months to progress throughout the body of a human. This is largely dependent on where an individual is infected, but when symptoms appear there is a 1% chance of survival. The reason that it is important to model the spread of rabies in foxes is because (1) it's potential to progress to humans by indirect methods such as infections from domesticated animals such as dogs and cats (2) The severity of the disease.

Being able to accurately model the spread of a disease is important as it can allow public health official to determine the best preventable course of action. The fox rabies SIR model was originally created in 1981 by Anderson, Jackson, May and Smith [1]. The original model is a system of ODEs which was used to map the spread of rabies across Europe.

$$
\begin{aligned}
\frac{\partial S}{\partial t} &= (aS - bS)\tfrac{1}{K} - \tfrac{(a-b)NS}{K} - \beta R \\
\frac{\partial I}{\partial t} &= -bI - \tfrac{(a-b)NI}{K} + \beta RS - \sigma I \\
\frac{\partial R}{\partial t} &= -bR - \tfrac{(a-b)NR}{K} + \sigma I - \alpha R
\end{aligned}
$$

The above equations are taken from [1]. SIR stands for Susceptible, Infected and Recovered however in rabies, because there is such a low chance of survival, SIR acronym is updated to Susceptible, Infectious and Rabid.

The SIR model divides the population into the three categories from the SIR acronym. Because this model is an ODE the only independent variable is (t)ime. By given a value of time the model can determine the number of foxes who are susceptible, infectious and rabid in the population.

The model was expanded upon in 1986 by Murray, Stanley and Brown [3] which added a two-dimensional spatial component to account for how rabid foxes make contact with non-rabid foxes.

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$$\frac{\partial S}{\partial t} = (aS - bS)\frac{1}{K} - \frac{(a-b)NS}{K} - \beta RS \tag{1}$$

$$\frac{\partial I}{\partial t} = -bI - \frac{(a-b)NI}{K} + \beta RS - \sigma I \tag{2}$$

$$\frac{\partial R}{\partial t} = -bR - \frac{(a-b)NR}{K} + \sigma I - \alpha R + D\Delta R \tag{3}$$

In equation (3), we can see that the Laplacian operator is added to the rabid PDE equation. When foxes become rabid they become aggressive and confused and this can lead to random wondering. This random wandering can be modeled by the diffusion term ($D\,\Delta R$) in the Rabid fox model from 1986.

$$D\,\Delta R = D\left(\frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2}\right). \tag{4}$$

2 Methodology

Equations (1)-(3) are a system of PDEs where we are given rates of change with respect to time for the Susceptible, Infected and Rabid population on the left hand side of the equation and the population density of the dependent variables on the right hand side. Several constants are included with the system of equations which are described below. The values were taken from [1].

Description	Symbol	Value
average birth rate	a	1 per year
average intrinsic death rate	b	0.5 per year
average duration of clinical disease	α	365/5
average incubation time	σ	365/28
carrying capacity	K_t	1 fox km^2
disease transmission coefficient	β	80 km^2 per year
carrying capacity	K	0.25 to 4.0 foxes km^2

Table 1: Parameter Values for Rabies SIR model

There are several assumptions made by the SIR model

1. The dynamics of the fox population in the absence of rabies can be approximated by the simple logistic law

$$\frac{dS}{dT} = (a-b)(1 - S/K)S \tag{5}$$

2. Rabies is transmitted from rabid to susceptible fox
3. Infected foxes become infectious at an average rate per head, $sigma$, where $1/\sigma$ is the average incubation time
4. Rabies is invariably fatal, with rabid foxes dying at an average per capita rate α
5. Rabid and infected foxes continue to put pressure on the environment and can die of causes other than rabies
6. Foxes are territorial and divide the countryside up into non-overlapping ranges
7. Rabies is transmitted by direct contact between foxes
8. Rabies acts on the central nervous system, including behavioural changes in the host.

Half of infected foxes have what might be called "furious rabies", and exhibit symptoms that are typically associated with the disease. These type of foxes may also become confused and lose their sense of direction. As mentioned in assumption 6, foxes are territorial. Rabid foxes, however, may wander randomly into different locations. The diffusion term, $D\Delta R$, in equation (3) is set to emulate random walking of foxes with rabies.

In order to solve the model we need a set initial conditions (listed below)

$$S(x,y,0) = \frac{[(\alpha+b)\beta K+(a-b)(\alpha+a)][\sigma\beta K(\sigma+b)+\alpha(a-b)(\sigma+a)]}{\beta^1[\sigma\beta K-a(a-b)]^2} \quad (6)$$

$$I(x,y,0) = \frac{[(\alpha+b)\beta K+(a-b)(\alpha+a)]R_0}{[\sigma\beta K-a(a-b)]} \quad (7)$$

$$R(x,y,0) = \frac{(a-b)[\sigma\beta K-(\sigma+a)(\alpha+a)]}{\beta[\sigma\beta K-a(-a)]} \quad (8)$$

The domain was chosen for $x\,(km) \in (-1000, 10000)$ and $y\,(km) \in (-1000, 1000)$ with time $T\,(years) \in (0, 20)$. Additionally the carrying capacity was tested with values from $0.9 \leq K \leq 4$. For K values less than 1 all models would not converge.

The SIR model equation, parameters and assumptions were from [3]. When trying to reproduce the results in [3] some modifications needed to be done.

2.1 Modifications

Originally the susceptible equation (1) had the form

$$\frac{dS}{dT} = aS - bS - \frac{(a-b)NS}{K} - \beta RS.$$

This equation would cause the population to not match the paper results. There would be no oscillation over time that should have occurred given some of the conditions of [3]

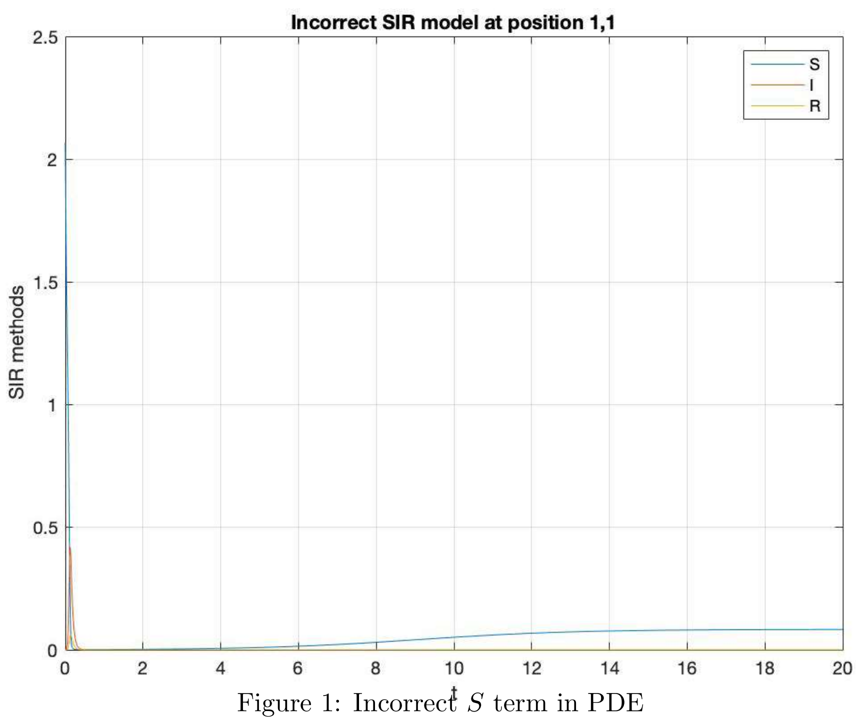

Figure 1: Incorrect S term in PDE

Figure (1) shows the results of this graph. The susceptible term was updated to match the equation from [3] (and found in equation (1)).

Additionally, it was found that by using the original parameters from [1] would break the model causing the population to explode over a few conditions. Updating to the newer values, listed below, allowed the model to work with a variety of ranges.

Parameter	1986 paper value	Updated value
α	5	365/5
σ	28	365/38

Table 2: Parameter updates

2.2 Numerical approximation methods

There are several methods for solving PDEs such as finite difference method, finite element and much more. In order to solve these types of systems one can update the PDE term and set it up as a system of ODEs. With the mimetic operator, *lap2D*, we were able to use the Laplacian operator in it's standard form.

With numerical methods we will create a grid of points which are evenly spaced apart and in the center of a grid. With the mimetic operators the the operations will be located as shown below.

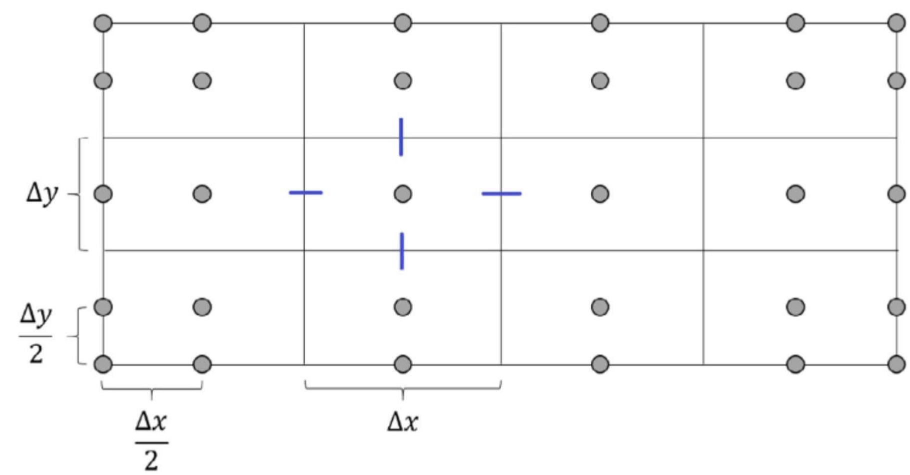

Figure 2: 2D Staggered grid taken from [2].

For the Laplacian, we create a sparse matrix created from the MOLE library. When multiplied by a vector of values the output is a vector which has the Laplacian applied to it.

Algorithm 1 Mimetic operator pseudocode

```
k = order of accuracy
m = number of grid cells in x dimension
dx = step size for x
n = number of grid cells in y dimension
dy = step size for y
L = lap2D(k,m, dx, n, dy)
U = 2D grid for x and y
U = reshape U to vector
U = L*U
U = reshape(U,m + 2, n + 2)
```

From Algorithm 1, we see that we state the parameters and grid dimensions for the mimetic 2D Laplacian operator. We also define a 2D matrix of values which we call U in this example. U contains the values at our grid point and can be considered a point in time. U is then reshaped to a matrix by concatenating its rows (because Matlab is column-major indexing). This new U vector is multiplied by the Laplacian operator which results in the Laplacian being applied to the U vector. After this U is reshaped back into a 2D x by y matrix.

Two methods were explored for numerical approximation of the time derivative. The first is the forward finite difference method. Forward finite different method can be implemented by using the discrete approximation of the a differential equation

$$\begin{aligned} \frac{\partial f}{\partial t} &= f(t) \\ \frac{f_{t+1} - f_t}{\Delta t} &= f(t) \\ f_{t+1} &= f(t) + \Delta t\, f(t) \end{aligned}$$

Given an initial condition at f_0 and having a known value for Δt allows us to numerically approximate the solution of the differential equation. The forward finite difference method is an explicit first order method. Because of this, it is of low accuracy it will often require a lower step size to approximate the solution. Given a spatial step size of $dx = 200, dy = 200$ it was found that the solution would not converge with a step size greater than $dt = 1e - 4$. This required significantly more time to converge to a solution with an average execution time of $6.38s$.

Algorithm 2 Forward finite difference with mimetic operator

```
num steps = 0:dt:T
S,I,R = zeros(m+2,n+2,length(num steps))
S(x,y,0) = S0, I(x,y,0) = I0, R(x,y,0) = R0
for ii = 0:num steps - 1 do
St = S(:,:,ii)
It = T(:,:,ii)
Rt = R(:,:,ii)
dS = (St,It,Rt) equation 1
dI = (St,It,Rt) equation 2
dR = (St,It,Rt) equation 3
S(:,:,ii+1) = St + dt dS
I(:,:,ii+1) = It + dt dI
R(:,:,ii+1) = Rt + dt dR
end for
```

The second method that was tested is the Runge-Kutta fourth-order method. This is a higher accurate explicit differencing method. Given that $f(t)$ is equal to the right hand side of the differential equation.

$$\begin{cases} kn_1 &= f_t \\ tmp_2 &= f_t + \frac{1}{2}\Delta t kn_1 \\ kn_2 &= f(tmp_2) \\ tmp_3 &= f_t + \frac{1}{2}\Delta t kn_2 \\ kn_3 &= f(tmp_3) \\ tmp_4 &= f_t + \Delta t kn_3 \\ f_{t+1} &= f_t + \frac{\Delta t}{6}(kn_1 + 2kn_2 + 2kn_3 + kn_4) \end{cases} \tag{9}$$

Equations (9) can be reconstructed in a similar fashion to Algorithm 2 to solve a system of equations. As shown with the above equations there is more computational complexity to this method. This method can be seen as taking the average of values between the step size in order to approximate a better solution. Although there is more computational complexity, we can use a larger step size in time while producing the same results. This results in a quicker execution time, on average with a step size of $dt = 0.01$ the algorithm would completed execution in $0.22s$.

While approximating both of these methods the 2D Laplacian mimetic operator with order of accuracy four was used in the (3) equation.

3 Results

The first method to review is the finite difference (FD) method for the time derivative. We found that at large step sizes (0.01) the forward finite-difference breaks down and is unable to properly model the system of PDEs. Being that FD is a first-order explicit method it has a small region of stability.

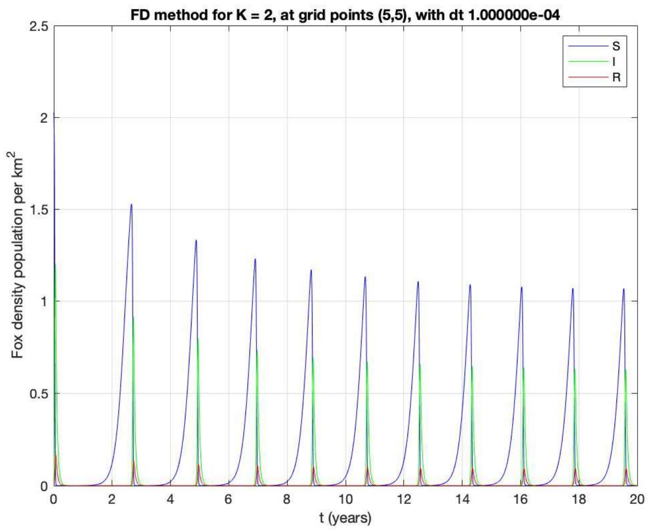

Figure 3: Forward difference in time: $dt = 0.0001$.

Compare the plot of forward difference (Figure 3) with the Runga-Kutta method (Figure 4) (which uses a step size $dt = 0.01$.

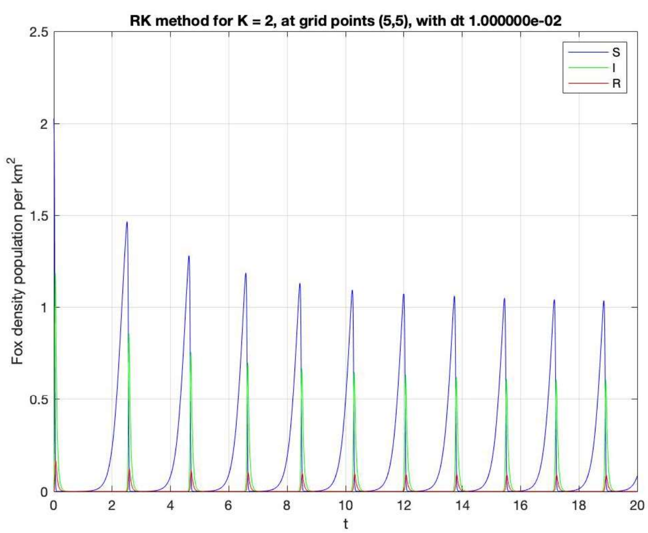

Figure 4: Fourth order Runge-Kutta in time: $dt = 1e - 2$.

Because RK4 method has higher accuracy with fewer steps, we will be using this method going forward. The below graph shows the waves of the population at different grid points for x and y.

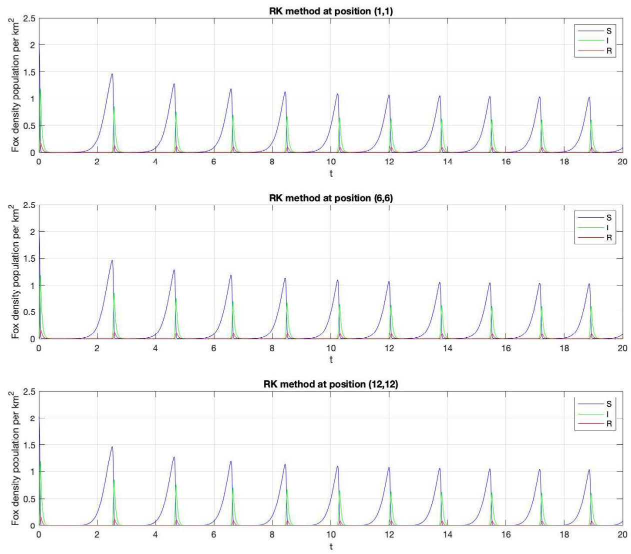

Figure 5: Various grid points at $K = 2$.

We can see that through time across the different points the distribution remains the same at each x, y point. This is also true for different values of K. Although we see from the below figure that the amplitude and speed of the wave has changed with $K = 1.1$.

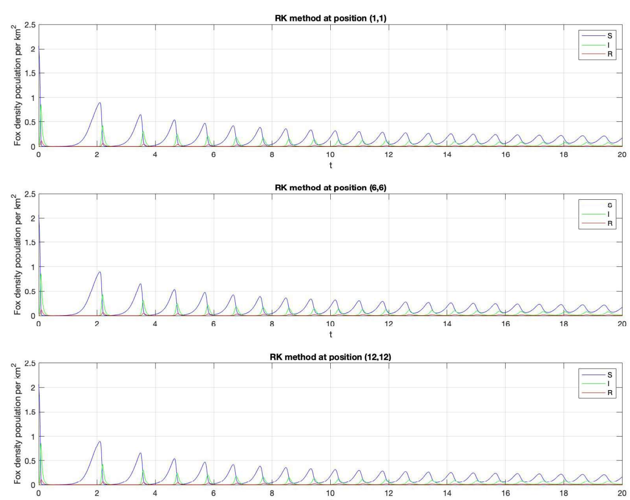

Figure 6: Various grid points at $K = 1.1$.

This makes sense as our initial condition evenly distributes the initial population densities at on all grids. Changing $K < 1$ however, breaks the model. This was tested with higher resolutions for dx, dy and dt but did not result in proper modeling of the solution.

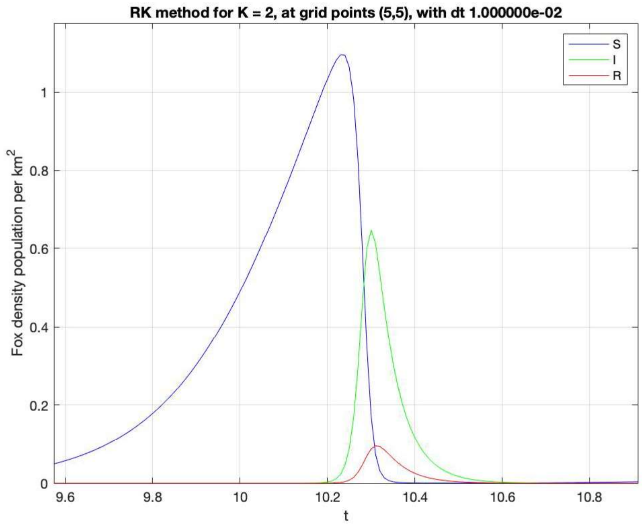

Figure 7: SIR ($K = 2$) at time 10.3 years.

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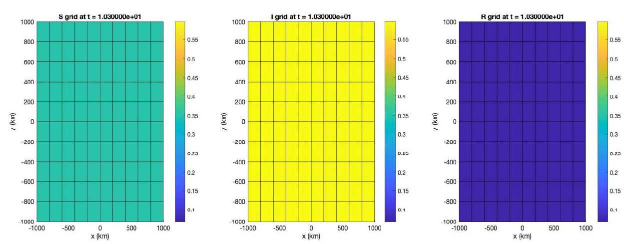

Figure 8: SIR ($K = 2$) grid view for S, I, R plots at time 10.3 years.

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[2] Corbino, J., and Castillo, J.E., MOLE: Mimetic Operators Library Enhanced: The Open-Source Library for Solving Partial Differential Equations using Mimetic Methods, 2017. `https://github.com/csrc-sdsu/mole`

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