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Energy Conservation of Second-Order Mimetic Difference Schemes for the 1D Advection Equation *

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Abstract

In this report, a proof of the energy conservation property of second-order mimetic difference schemes is demonstrated for the one-dimensional advection PDE. This proof leverages on the discrete analog of the integration by parts mimetic difference property.

1 Introduction

Mimetic difference schemes for structured staggered grids have been around for a couple of decades [1] [2] [3]. To be able to replicate physical properties and conservation laws, discrete versions of scalar fields are represented at cell centers (include points on the boundary of the domain), while discrete versions of vector fields are characterized at grid faces. These faces, for one-dimensional domains are points; for two-dimensional domains are edges; for three-dimensional domains proper faces; and so on.

Mimetic difference schemes create analogs of differential operators (G for the gradient, D for the divergence, C for the curl, L for the Laplacian), that not only preserve their vector calculus identities but also hold discrete counterparts of integral formulas. The latter is triggered by the introduction of a boundary operator B and weighted norms P and Q , (for the gradient and divergence operators in the integrand, respectively) to enforce the additional goal of constant high-order accuracy on the whole staggered grid including boundary cells.

Energy conservation of mimetic differences have been found in numerical applications but no targeted effort has been attempted up to date to prove this theoretical property of these methods.

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Perhaps two of the reasons has been the difficulty of writing the vector calculus integral identities explicitly in two and three dimensions as well as the lack of appropriate interpolation operators that have constant high-order accuracy over the whole staggered domain.

The proof is restricted to second-order mimetic difference divergence operators, but it can be easily generalized to second-order gradient operators.

This work is the basis for demonstrating energy conservation of mimetic difference schemes in general.

2 Proof

As mentioned in the introduction, in this report, we keep constrained to one-dimensional dimensional to avoid some of the multi-dimensional hurdles that complicate the proof of the energy conservation properties of the mimetic difference schemes.

Consider the following one-dimensional advection partial differential equation

$$u_t + u_x = 0, \quad x \in (-1, 1), \quad t > 0, \quad (1)$$

$$u(-1, t) = g(t), \quad t > 0, \quad (2)$$

$$u(x, 0) = u_0(x), \quad (3)$$

with condition (2) on the left boundary, and initial condition (3).

It is well-known that (1)-(3) is a well-posed PDE and that its discontinuous Galerkin Spectral Element Method discretization has the energy conservation property [4].

By multiplying (1) by u and integrating over the spatial domain, one obtains

$$\int_{-1}^1 u u_t dx + \int_{-1}^1 u \nabla \cdot u dx = 0, \quad (4)$$

where the spatial derivative has been considered as a divergence.

The first and second terms in (4), respectively hold,

$$\begin{aligned} \int_{-1}^1 u u_t dx &= \int_{-1}^1 \frac{1}{2} \frac{du^2}{dt} dx = \frac{1}{2} \frac{d}{dt} \left(\int_{-1}^1 u^2 dx \right), \\ \int_{-1}^1 u \nabla \cdot u dx &= \frac{1}{2} \int_{-1}^1 \nabla \cdot (u^2) dx, \end{aligned}$$

and hence, after a time integration from 0 to T , (4) becomes

$$\left(\int_{-1}^1 (u^2(x, T) - u^2(x, 0)) dx \right) + \int_0^T \int_{-1}^1 \nabla \cdot (u^2) dx dt = 0. \quad (5)$$

Now, we plan write discrete analogs of (5) according to mimetic difference schemes leveraging on properties of the formulation of the integration by parts principle [1] [2] [3].

Define vector U , a mimetic numerical approximation of $u(x, t)$, on the staggered grid

$$-1 = x_0, x_{\frac{1}{2}}, \dots, x_{N-\frac{1}{2}}, x_{N+1} = 1,$$

with

$$x_{j-\frac{1}{2}} = -1 + (j - \frac{1}{2})h, h = \frac{1}{N}, j = 1, \dots, N,$$

where the cell centers are equally separated by $h = \frac{1}{N}$. So,

$$U = (U(x_0, t), U(x_1, t), \dots, U(x_N, t), U(x_{N+1}, t))^T.$$

For functions defined on the staggered grid, the mimetic difference schemes use the second-order composite midpoint rule for integration

$$\int_{-1}^1 v(x) dx \approx h \sum_{i=0}^{N-1} v_{i+\frac{1}{2}}, \quad v_{i+\frac{1}{2}} = v(x_{i+\frac{1}{2}}), \quad i = 1, \dots, N.$$

The presence of the first integral of (5) motivates the definition of (discrete) energy of U at time t by

$$E(t) = h \sum_{i=0}^{N-1} U^2(x_{i+\frac{1}{2}}, t).$$

Applying the composite midpoint rule for the first term and utilizing for the second term the weighted inner product for the divergence, (5) becomes

$$h \sum_{i=0}^{N-1} U^2(x_{i+\frac{1}{2}}, T) + \frac{1}{h} \int_0^T \mathbf{1} Q D I_D U^2 dt = h \sum_{i=0}^{N-1} U^2(x_{i+\frac{1}{2}}, 0), \quad (6)$$

where I_D is the second-order interpolation operator from the staggered cell centers to nodes of [5], or equivalently, the usual linear interpolation.

The mimetic form of the fundamental theorem of calculus for the divergence states that

$$\begin{aligned} \mathbf{1} Q D (I_D U^2) &= (-1, 0, \dots, 0, 1)(U^2(x_0, t), U^2(x_1, t), \dots, U^2(x_{N-1}, t), U^2(x_N, t))^T \\ &= -U^2(-1, t) + U^2(1, t) \end{aligned}$$

and hence (6) becomes,

$$h \sum_{i=0}^{N-1} U^2(x_{i+\frac{1}{2}}, T) + \frac{1}{h} \int_0^T U^2(1, t) dt = h \sum_{i=0}^{N-1} U^2(x_{i+\frac{1}{2}}, 0) + \frac{1}{h} \int_0^T U^2(-1, t) dt$$

Using the boundary condition (2), and the discrete energy definition, one verifies that

$$E(T) + \frac{1}{h} \int_0^T U^2(1, t) dt = E(0) + \frac{1}{h} \int_0^T g^2(t) dt,$$

i.e., the energy at T plus the energy lost at the right boundary matches the initial energy plus the energy gained at the left boundary.

References

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