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Interpolation Operators for Staggered Grids *

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Abstract

In this report, a derivation of interpolation operators for scalar and vector fields on staggered grids is shown. This technique avoids the inversion of a Vandermonde matrix and provides a faster and more accurate computation of interpolation operators.

1 Introduction

Gradient G and divergence D mimetic difference operators in d dimensions are defined on staggered grids.

If the staggered grid contains m_i cells along the *i*-th axis, $i = 1, \dots, d$, and the set of staggered grid cell center points in the *i*-th direction, including its boundaries, is given by C_i (with cardinality $m_i + 2$) then the collection of grid cell center points is given by $C = C_1 \times \cdots \times C_d$ and its cardinality is given by $N_c = \prod_{i=1}^d (m_i + 2)$.

On the other hand, the set F_i of d-faces (for d=1 points, for d=2 edges, for d=3 faces, and so on), associated to the i-th component of a vector field, of the staggered grid along the i-axis, has a cardinality given by $N_{x_i} = m_i \Pi_{j \neq i}(m_j + 1)$ and hence the set of all d-dimensional faces of the staggered grid $F = \bigcup_{i=1}^d F_i$, has a cardinality of $N_f = \sum_{i=1}^d N_{x_i} = \sum_{i=1}^d m_i \Pi_{j \neq i}(m_j + 1)$.

Like its continuous counterpart, mimetic gradient operators are applied on scalar fields, which usually are defined on C. Mimetic gradient operators return spatial partial derivatives at F. In particular, 2D mimetic gradient operators (in coordinates x and y) are split in two components $[G_x, G_y]^T$, the first of which, when applied to a scalar field $v = (v_1, v_2)$, with $v_1 : F_x \to \mathbb{R}^2$, $v_2 : F_y \to \mathbb{R}^2$, provides the value of $\frac{\partial v_1}{\partial x}$ evaluated at F_x . Similarly, the second component of the 2D mimetic gradient operator computes $\frac{\partial v_2}{\partial y}$ evaluated at F_y .

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On the other hand, 2D mimetic divergence operators $D = [D_x, D_y]$ act, onto vector field components v_1 and v_2 and compute $\frac{\partial v_1}{\partial x}, \frac{\partial v_2}{\partial y}$, at F_x and F_y , respectively.

It may happen that after applying one of the mimetic operators at a scalar or vector field, the result should be the input of another mimetic operator and the data might not be at the appropriate points. In that case, some interpolation might be required.

2 Form of the interpolation operators for d > 1 dimensions

One can foresee that two interpolation operators are required, independently of the space dimensionality of the PDE to be solved and of the level of accuracy of the operators. One operator should interpolate data from cell centers to cell faces, namely $I_X^{c \to f}: C \to F$ with $X = (x_1, \dots, x_d)$, and another from cell faces to cell centers, namely, $I_X^{f \to c}: F \to C$. The subindex X in both interpolation operators refer to the number of dimensions.

The fact that interpolation operations I_X (where I_X refers to $I_X^{c \to f}$ or $I_X^{f \to c}$) in d-dimensions, with d > 1 should apply separately on each dimension, imposes the constraint that they should be of the form

where $I_{X,i}$, $i = 1, \dots, d$, applies to the *i*-th component of the data.

Since the gradient operator in d-dimensions, d > 1, is of the form

$$G_{x_1\cdots x_d} = \begin{bmatrix} \hat{I}_{m_d}^T \otimes \hat{I}_{m_{d-1}}^T \otimes \cdots \otimes \hat{I}_{m_2}^T \otimes G_{x_1} \\ \hat{I}_{m_d}^T \otimes \cdots \otimes \hat{I}_{m_3}^T \otimes G_{x_2} \otimes \hat{I}_{m_1}^T \\ & \vdots \\ G_{x_d} \otimes \hat{I}_{m_{d-1}}^T \otimes \cdots \otimes \hat{I}_{m_2}^T \otimes \hat{I}_{m_1}^T \end{bmatrix},$$

where

$$\hat{I}_p = [0_{p \times 1} | I_p | 0_{p \times 1}]^T, \quad I_p = diag\{1, \dots, 1\},$$

and with its data is on C and its image is evaluated at F, then $I_X^{c \to f}$ structure has to be of the form

where $I_{X,i}^{c \to f}$, $i = 1, \dots, d$, is the one-dimensional interpolation operator from cell centers to cell faces in the *i*-th dimension.

Notice that because of the form of $I_X^{c \to f}$ and $Gx_1 \cdots x_d$, the d-dimensional cell centers to cell faces interpolation operator should act on the image of the d-dimensional gradient operator.

Similarly, since the d-dimensional divergence operator is of the form

$$D_{x_1\cdots x_d} = \left[\hat{I}_{m_d} \otimes \cdots \otimes \hat{I}_{m_2} \otimes D_{x_1}, \quad \hat{I}_{m_d} \otimes \cdots \otimes \hat{I}_{m_3} \otimes D_{x_2} \otimes \hat{I}_{m_1}, \quad \cdots, \quad D_{x_d} \otimes \hat{I}_{m_{d-1}} \otimes \cdots \otimes \hat{I}_{m_1} \right],$$

and maps data from the cell faces to the cell centers, then the interpolation operator $I_X^{f \to c}$ should be of the form

$$I_{X}^{f \to c} = \begin{bmatrix} \hat{I}_{m_{d}} \otimes \cdots \otimes \hat{I}_{m_{2}} \otimes I_{X,1}^{f \to c} & & & & \\ & \hat{I}_{m_{d}} \otimes \cdots \otimes \hat{I}_{m_{3}} \otimes I_{X,2}^{f \to c} \otimes \hat{I}_{m_{1}} & & & & \\ & & & & \ddots & & \\ & & & & & I_{X,d}^{f \to c} \otimes \hat{I}_{m_{d-1}} \otimes \cdots \otimes \hat{I}_{m_{1}} \end{bmatrix},$$

where $I_{X,i}^{f\to c}$, $i=1,\cdots,d$, is the one-dimensional interpolation operator from cell faces to cell centers in the *i*-th dimension.

Notice that because of the form of $I_X^{f\to c}$ and $Dx_1\cdots x_d$, the d-dimensional divergence operator should act on the image of the d-dimensional cell faces to cell centers interpolation operator.

What remains now is to determine the one-dimensional interpolation operators.

3 The one dimensional interpolation operators

A motivation for writing this report was [2]. In that document, interpolation operators for the divergence and gradient mimetic operators are obtained for orders k = 2, 4, 6. They are derived via the inversion of a Vandermonde matrix. ¹

Since the interpolation operators are more related to geometric entities that to differential operators, we introduce a more appropriate notation for them. To facilitate their relationship with the interpolation operators defined in [2], when defining the interpolation operators both notations are kept.

In [1], an exact algorithm for the computation of the inverse of the Vandermonde matrix is given. As we shown below, it is possible to extract the first row of it (which is actually what is needed for the interpolation operators) and find $I_x^{c \to f}$ and $I_x^{f \to c}$ (the equivalents to

¹We noticed that order k = 8 was missing because of stability issues of the direct inversion.

the divergence I_D^k and gradient I_G^k interpolation operators of order k in [2]), for all orders k = 2, 4, 6, 8, exactly.

From [1], one can infer that if the generator of the Vandermonde matrix is given by

$$g = [c_1, \cdots, c_m]',$$

then the first row of the inverse of the respective Vandermode matrix (which corresponds to the row of interest of the interpolator operator) is given by

$$\left[\frac{p_1}{d_1}, \cdots, \frac{p_m}{d_m}\right],\,$$

where

$$p_i = \frac{p}{c_i}, \qquad p = \prod_{i=1}^m c_i, \qquad d_i = \prod_{j \neq i} (c_j - c_i).$$

3.1 Cell centers to cell faces one-dimensional interpolation operators

The cell centers to cell faces one-dimensional interpolation operators are matrices of order $(N+1) \times (N+2)$ with N the number of cells. They can be obtained without inverting explicitly the Vandermonde matrix with g generator.

For accuracy order k=2, the generator of the cell centers to cell faces interpolator $I_x^{c\to f,(2)}$ is

$$d_2 = \left[-\frac{1}{2}, \frac{1}{2} \right]',$$

and

$$I_x^{c o f,(2)} = I_D^2 = rac{1}{2} \left[egin{array}{ccccccc} 2 & & & & & \\ & 1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & 1 & \\ & & & & 2 \end{array}
ight].$$

For accuracy order k=4, the generator for the interior scheme of the cell centers to cell faces interpolator $I_x^{c\to f,(4)}$ is

$$d_4 = \left[-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \right]'.$$

In addition, the generator associated to the interpolation near the boundary is

$$d_{41} = \left[-1, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \right]'.$$

Thus, $I_x^{c \to f,(4)}$ is given by

$$I_x^{c \to f, (4)} = I_D^4 = \frac{1}{112} \begin{bmatrix} 112 \\ -16 & 70 & 70 & -14 & 2 \\ & -7 & 63 & 63 & -7 \\ & & \ddots & \ddots & \ddots \\ & & & -7 & 63 & 63 & -7 \\ & & & 2 & -14 & 70 & 70 & -16 \\ & & & & & 112 \end{bmatrix}.$$

For accuracy k=6, the generator for the interior scheme of the cell centers to cell faces interpolator $I_x^{c\to f,(6)}$ is

$$d_6 = \left[-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \right]'.$$

In addition, the generators associated to the interpolation near the boundary are

$$d_{61} = \left[-1, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2} \right]',$$

$$d_{62} = \left[-2, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2} \right]'.$$

So, $I_x^{c \to f,(6)}$ is given by

$$I_x^{c \to f, (6)} = I_D^6 = \frac{1}{8448} \begin{bmatrix} 8448 \\ -768 & 4158 & 6930 & -2772 & 1188 & -330 & 42 \\ 256 & -924 & 4620 & 5544 & -1320 & 308 & -36 \\ 99 & -825 & 4950 & 4950 & -825 & 99 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & 99 & -825 & 4950 & 4950 & -825 & 99 \\ & & & & -36 & 308 & -1320 & 5544 & 4620 & -924 & 256 \\ & & & & & 42 & -330 & 1188 & -2772 & 6930 & 4158 & -768 \\ & & & & & & & 8448 \end{bmatrix}$$

For accuracy order k = 8, the generator for the interior scheme of the cell centers to cell faces interpolator $I_x^{c \to f,(8)}$ is

$$d_8 = \left[-\frac{7}{2}, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2} \right]'$$

The generators associated to the interpolation near the boundary are

$$d_{81} = \left[-1, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2} \right]',$$

$$d_{82} = \left[-2, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2} \right]',$$

$$d_{83} = \left[-3, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2} \right]'.$$

Hence, $I_x^{c \to f,(8)}$ is given by

$$\begin{bmatrix} 1 \\ -\frac{1}{15} & \frac{429}{1024} & \frac{1001}{1024} & -\frac{3003}{5120} & \frac{429}{1024} & -\frac{715}{3072} & \frac{91}{1024} & -\frac{21}{1024} & \frac{11}{5120} \\ \frac{1}{65} & -\frac{33}{512} & \frac{231}{512} & \frac{2079}{2560} & -\frac{165}{512} & \frac{77}{512} & -\frac{27}{512} & \frac{77}{6656} & -\frac{3}{2560} \\ -\frac{1}{143} & \frac{27}{1024} & -\frac{105}{1024} & \frac{567}{1024} & \frac{675}{1024} & -\frac{175}{1024} & \frac{567}{11264} & -\frac{135}{13312} & \frac{1}{1024} \\ -\frac{5}{2048} & \frac{49}{2048} & -\frac{245}{2048} & \frac{1225}{2048} & \frac{1225}{2048} & -\frac{245}{2048} & \frac{49}{2048} & -\frac{5}{2048} \\ & \ddots \\ -\frac{5}{2048} & \frac{49}{2048} & -\frac{245}{2048} & \frac{1225}{2048} & \frac{1225}{2048} & \frac{1225}{2048} & \frac{1225}{2048} & \frac{245}{2048} & -\frac{245}{2048} & \frac{49}{2048} & -\frac{5}{2048} \\ & & & & & & & & & & & & & & & \\ & & & & & & & & & & & & & & \\ & & & & & & & & & & & & & & \\ & & & & & & & & & & & & & \\ & & & & & & & & & & & & \\ & & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\$$

3.2 Cell faces to cell centers one-dimensional interpolation operators

Cell faces to cell centers one-dimensional interpolation operators are matrices of order $(N+2)\times(N+1)$ with N the number of cells. They can be obtained without inverting explicitly the Vandermonde matrix with g generator.

For accuracy k=2, the generator of the cell faces to cell centers interpolator $I_x^{f\to c,(2)}$ is

$$g_2 = \left[-\frac{1}{2}, \frac{1}{2} \right]'$$

and

$$I_x^{f o c,(2)} = I_G^2 = rac{1}{2} \left[egin{array}{cccc} 2 & & & & & \\ 1 & 1 & & & & \\ & \ddots & \ddots & & & \\ & & & 1 & 1 & \\ & & & & 2 & \end{array}
ight].$$

For accuracy k=4, the generator for the interior scheme of the cell faces to cell centers interpolator $I_x^{f\to c,(4)}$ is

$$g_4 = \left[-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \right]'.$$

The generator associated to the interpolation near the boundary is

$$g_{41} = \left[-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2} \right]'.$$

Thus, $I_x^{f\to c,(4)}$ is given by

$$I_x^{f \to c, (4)} = I_G^4 = \frac{1}{128} \begin{bmatrix} 128 \\ 35 & 140 & -70 & 28 & -5 \\ -8 & 72 & 72 & -8 \\ & \ddots & \ddots & \ddots & \ddots \\ & & -8 & 72 & 72 & -8 \\ & & -5 & 28 & -70 & 140 & 35 \\ & & & & & 128 \end{bmatrix}.$$

For accuracy k=6, the generator for the interior scheme of the cell faces to cell centers interpolator $I_x^{f\to c,(6)}$ is

$$g_6 = \left[-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \right]'.$$

The generators associated to the interpolation near the boundary are

$$g_{61} = \left[-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2} \right]',$$

$$g_{62} = \left[-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2} \right]'.$$

So, $I_x^{f \to c,(6)}$ is given by

For accuracy k=8, the generator for the interior scheme of the cell faces to cell centers interpolator $I_x^{f\to c,(8)}$ is

$$g_8 = \left[-\frac{7}{2}, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2} \right]'.$$

The generator associated to the interpolation near the boundary are

$$g_{81} = \left[-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \frac{15}{2} \right]',$$

$$g_{82} = \left[-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2} \right]',$$

$$g_{83} = \left[-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2} \right]'.$$

Hence, $I_x^{f \to c,(8)}$ is given by

References

- [1] Rawashdeh E.A., A simple method for finding the inverse of a Vandermonde matrix, Matematički Vesnik, September 2019.
- [2] Srinivasan, A., Corbino, J., and Castillo, J.E., Implementation Of Mimetic Relaxation Runge Kutta Methods, CSRC Report, 3-Jan-2022.