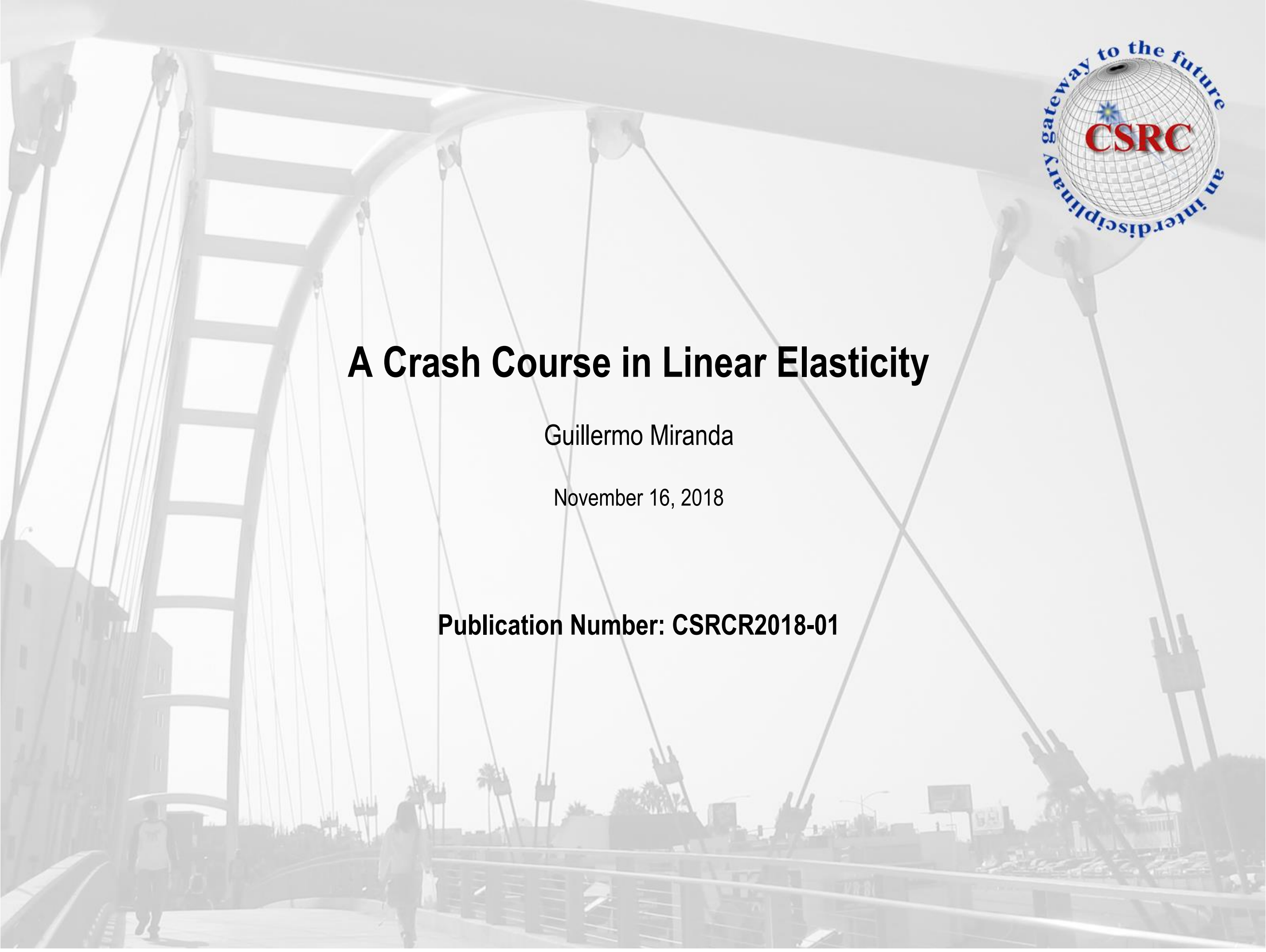

A Crash Course in Linear Elasticity

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A Crash Course in Linear Elasticity

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Abstract

The present report is guided by the idea of providing Graduate students at CSRC, not having a previous technical knowledge of the physics underlying the momentum PDE's describing deformations and stresses in solid media under Hooke's linear stress-strain relations, with a clear and intuitive presentation, that is analytically rigorous at the same time. This presentation is combined with some detailed explanation of the numerical solutions of elastic vibrations that can be both time–stable and uniformly accurate in 3-D space up to the boundaries, by means of the combined use of High–Order Castillo–Grone mimetic finite difference operators on staggered grids and of symplectic stepping up in time.

1. An intuitive Perception of "Shear" and "Normal" Stresses.

Consider a long prismatic elastic bar with rectangular or square cross section with negligible weight horizontally resting undeformed upon supporting structure, and let us apply a load in the form of a concentrated vertical force $\vec{P}$, exactly at the bar top center, so that two equal reaction forces $(+\frac{P}{2}\hat{\boldsymbol{k}})$ develop at the surface contact of the supporting structure with the now pressing downwards bar.

Imagine that the bar is composed of very thin long elastic fibers, capable of elongating or contracting under longitudinally applied deformation forces. Due to the resulting deformed bar shape, top fibers (T) will contract, while bottom fibers (B) will elongate, and there will be just one fiber, the "neutral" one (N) placed midway between top and bottom fibers, that will keep its original length in the bar, prior to applying force $(\vec{P} = -P\hat{\boldsymbol{k}})$:

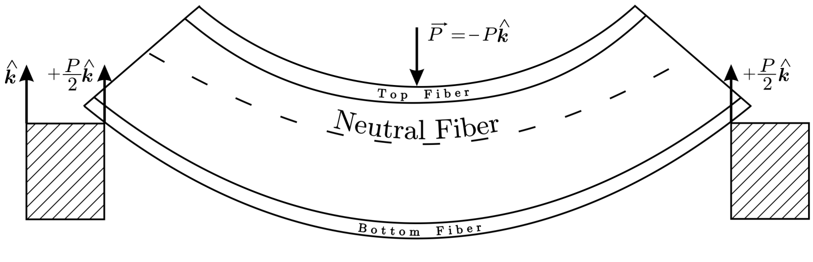


Figure 1. Engineering description: "simply supported beam".

1.1. Engineering Description: "Simply Supported Beam".

A usual visual approximation neglects the curvature deformation, and **the simplified model**, with the origin of cartesian coordinates x, y, z located at the center of the "neutral" or mid fiber **looks like** (This is physically impossible though, bur it helps understanding stress signs):

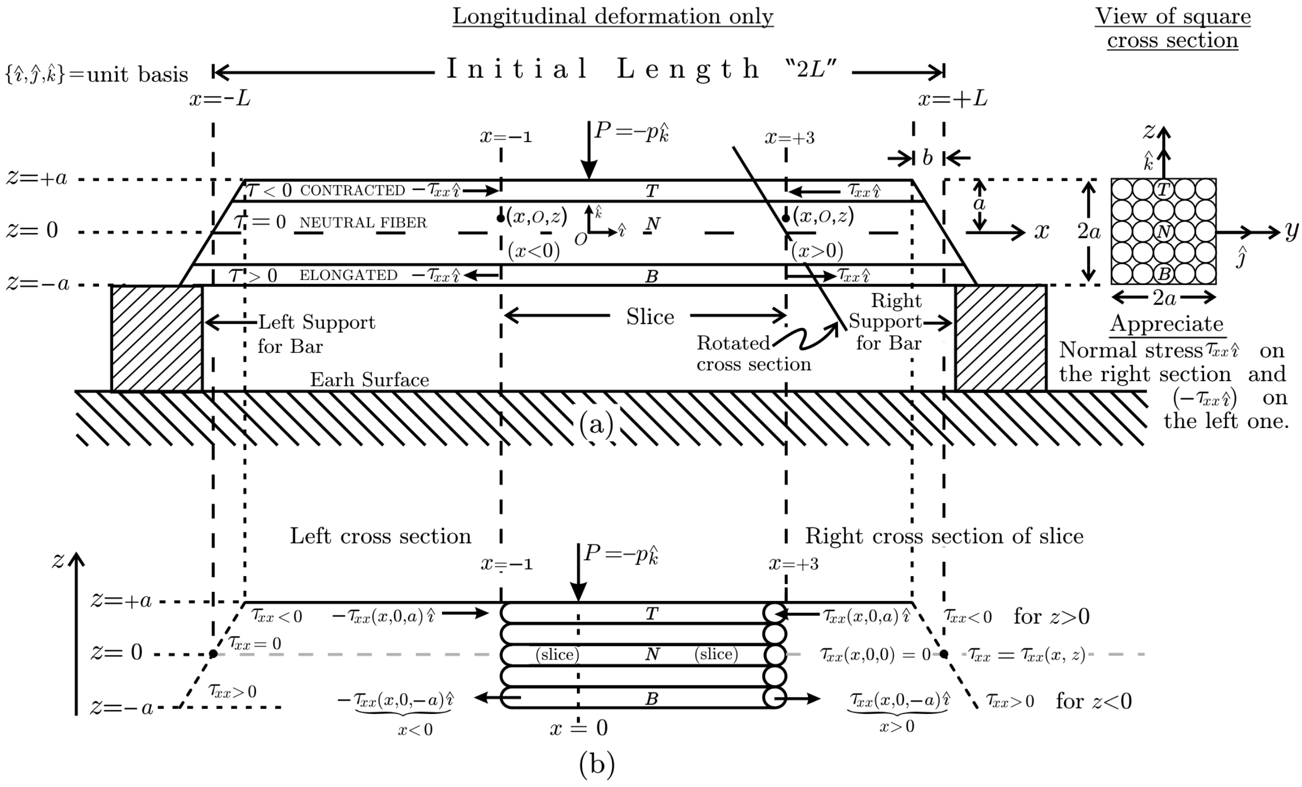


Figure 2. (a) Simplified non curved deformation of elastic beam fibers and (b) Elastic slab exhibiting τ_{xx}.

Simplest way to visualize **stress components $\boldsymbol{\mathcal{T}}_{ij}$ of stress tensor $\boldsymbol{\mathcal{T}}$**. Consider a right cross section of the horizontal bar at $x > 0$, normal $\hat{\boldsymbol{\imath}}$. The left part of the bar after cross section at x shows, on right section, $\mathcal{T}_{xx}\hat{\boldsymbol{\imath}}$, $\mathcal{T}_{xy}\hat{\boldsymbol{\jmath}}$, $\mathcal{T}_{xz}\hat{\boldsymbol{k}}$ over surface area $\Delta y\Delta z$ (see Fig. B3 App B in [1]) also, center ($\Delta y\Delta z$) at "N" ($z = 0$, $y = 0$) (Neutral Fiber):

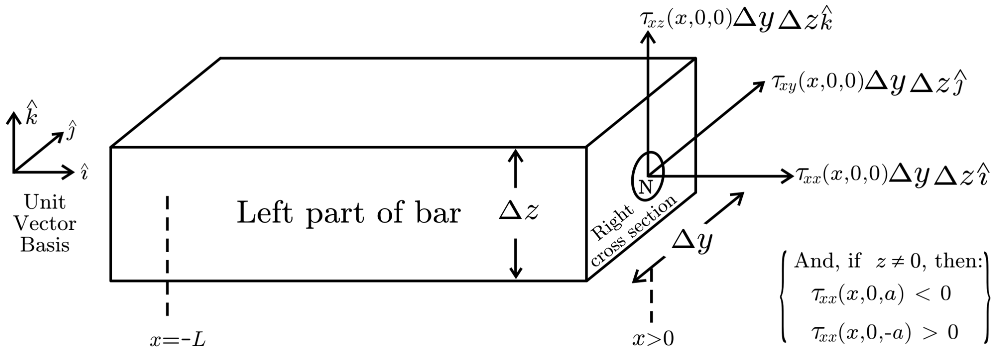


Figure 3. Cross-section of bar exhibiting stress components.

The reason to introduce "slices" in 3D bodies, is that we only see or measure directly **external forces or stresses** applied to the boundary surface of the whole body. We know that under deformation from an equilibrium state, elastic solids develop **Internal Stresses**, but we cannot see them or measure directly. Therefore, scientists have imagined that these internal forces, resulting from adding the internal stresses over some internal imaginary surface or **section**, must be such **as to impede the body thus "partitioned" or "sectioned"**, from falling appart into broken pieces or parts.

Thus, if we image a **mental slicing** of the 3D Body, we must **place** in both separated surfaces then appearing, substitute **stresses** that will allow a conservation of the body as such, and not really divided into pieces. **It is a useful mental device only.**

Simple Label Rule (Lettered Indexation):

Remember:

- **The First Letter** (x in the example) in $\mathcal{T}$, signals the direction of the unit normal ($\hat{\boldsymbol{\imath}}$ in this case) to the surface element with area dS, with differential lengths Labeled with the other two letters (y and z here),
- **The second letter in $\mathcal{T}$ signals** the direction of the corresponding tangential components:

$$\mathcal{T}_{xy}, \ \text{"}y\text{" goes with } \hat{\boldsymbol{\jmath}}, \qquad \mathcal{T}_{xz}, \ \text{"}z\text{" goes with } \hat{\boldsymbol{k}}.$$

Equivalent Index Notation: $x \longleftrightarrow 1, \quad y \longleftrightarrow 2, \quad z \longleftrightarrow 3$:

$$\mathcal{T}_{11} = \mathcal{T}_{xx}, \qquad \mathcal{T}_{12} = \mathcal{T}_{xy}, \qquad \mathcal{T}_{13} = \mathcal{T}_{xz}.$$

Same procedure for cross sections with normales $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$, completing visualization of stress tensor $\mathcal{T} = \{\tau_{ij}\}$.

Cartesian Strain Tensor

$$\varepsilon = \{\varepsilon_{ij}\}. \qquad (Also 1 = x, \ 2 = y, \ 3 = z).$$

For the elastic displacement vector

$$\vec{u} = u_1 \hat{\boldsymbol{\imath}} + u_2 \hat{\boldsymbol{\jmath}} + u_3 \hat{\boldsymbol{k}}, \quad \text{(index notation)}$$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \qquad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \qquad \varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right), \quad \text{etc.}$$

Defining

$$\mathrm{Tr}(\varepsilon) = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33},$$

Isotropic Hooke's Law is:

$$\boxed{\mathcal{T} = \lambda \mathrm{Tr}(\varepsilon) I + 2\mu\varepsilon},$$

where

$$I = \{\delta_{ij}\}, \quad \delta_{11} = \delta_{22} = \delta_{33} = 1, \quad \delta_{ij} = 0, \quad \text{when} \quad i \neq j.$$

The elastic constants **λ and μ are called Lamé's first and second parameters**. From experimental physics, other elastic constants appear naturally, but all of them can be expressed in terms of any two of them. A conversion table for the most common is:

- **Young Modulus:**

$$E = \frac{\mu}{\lambda + \mu}(3\lambda + 2\mu) = 2G(1 + \nu) = 3K(1 - 2\nu) \tag{1}$$

- **Shear Modulus:**

$$G = \mu = \frac{E}{2(1+\nu)} = \tfrac{3}{2}K\frac{1-2\nu}{1+\nu} \tag{2}$$

- **Poisson's Coefficient:**

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \tag{3}$$

- **Bulk Modulus:**

$$K = \lambda + \tfrac{2}{3}\mu = \frac{E}{3(1-2\nu)} \tag{4}$$

E and ν appear naturally together in strains ε_{xx}, ε_{yy}, ε_{zz},

$$\varepsilon_{xx} = \tfrac{1}{E}\left[\mathcal{T}_{xx} - \nu\left(\mathcal{T}_{yy} + \mathcal{T}_{zz}\right)\right], \quad \text{etc.} \tag{5}$$

Acceleration and rotational motions of infinitesimal rectangular parallelepiped. From Figure B.3 (see also the Figure in Section 4) in Appendix B (Castillo-Miranda's Book [1]) in absence of body forces $\vec{b}$ get, for $\vec{u} = u_1\hat{\boldsymbol{\imath}} + u_2\hat{\boldsymbol{\jmath}} + u_3\hat{\boldsymbol{k}}$:

$$\rho\left\langle \frac{\partial^2 \vec{u}}{\partial t^2}, \hat{\boldsymbol{k}} \right\rangle = \frac{\partial \mathcal{T}_{zx}}{\partial x} + \frac{\partial \mathcal{T}_{zy}}{\partial y} + \frac{\partial \mathcal{T}_{zz}}{\partial z} = \rho \frac{\partial^2 u_3}{\partial t^2}, \qquad \text{etc} \tag{6}$$

Equation (6) corresponds to equation (B.27) in [1], but it is **not the original equation** (B.26), which is the one really obtained from Figure B.3, namely,

$$\frac{\partial \mathcal{T}_{xz}}{\partial x} + \frac{\partial \mathcal{T}_{yz}}{\partial y} + \frac{\partial \mathcal{T}_{zz}}{\partial z} = \rho\left\langle a, \hat{\boldsymbol{k}} \right\rangle$$

and which took into consideration the tensional forces

$$\mathcal{T}_{xz}(x + dx,\, y,\, z) dy\, dz \hat{\boldsymbol{k}} \qquad \text{and} \qquad \mathcal{T}_{xz}(x,\, y,\, z) dy\, dz(-\hat{\boldsymbol{k}}),$$

acting on opposite parallel planes orthogonal to the Ox axis, plus the the pair of forces for $\mathcal{T}_{yz}$ and the pair of forces for $\mathcal{T}_{zz}$ acting on their planes, but all acting along the Oz axis. Observe that the limiting process $dx\, dy\, dz$ tending to zero induces a dimensional "collapse" of the finite sized rectangular parallelepiped into the single vertex point $P(x,\, y,\, z)$, in such a manner that tensions $\mathcal{T}$ that were **not** originally acting on the same points or planes, now appear evaluated at the same point, but through different planes with unit normals $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$. The change made possible by the symmetry of the stress tensor, allowing us to change the original $\mathcal{T}_{xz}$, $\mathcal{T}_{yz}$ and $\mathcal{T}_{zz}$ of equation (B.26) into $\mathcal{T}_{zx}$, $\mathcal{T}_{zy}$ and $\mathcal{T}_{zz}$ of the final equation (B.27), is now conveniently expressed in terms of the three components of the physically meaningful force per unit area

$$\vec{t}_z = \mathcal{T}_{zx}\hat{\boldsymbol{\imath}} + \mathcal{T}_{zy}\hat{\boldsymbol{\jmath}} + \mathcal{T}_{zz}\hat{\boldsymbol{k}}$$

(formula (B.10) in [1]). Thus, even though while working with the Fry–Richardson approach, considering finite sized cells and not points for numerical schemes using staggered 3D–grids, as in Castillo–Grone mimetic difference operators, this symmetry property of the stress tensor, allowing us to discretize (B.27) instead of (B.26), is then keeping the storage of tensions faithful to the physical meaning of the individual scalar components of the vector tensions like $\vec{t}_z$ etc.

Rotational motion about rotation axis $\boldsymbol{Oz}$, $\boldsymbol{Oy}$, $\boldsymbol{Ox}$:

$$\left.\begin{array}{l} \mathcal{T}_{xy} = \mathcal{T}_{yx} \\ \mathcal{T}_{xz} = \mathcal{T}_{zx} \\ \mathcal{T}_{yz} = \mathcal{T}_{zy} \end{array}\right\} \quad \text{or, in index notation} \quad \mathcal{T}_{ij} = \mathcal{T}_{ji} \quad \text{when } i \neq j.$$

The symmetry of the stress tensor $\mathcal{T} = \{\mathcal{T}_{ij}\}$, induces a corresponding symmetry for the strain tensor, that is, $\varepsilon_{ij} = \varepsilon_{ji}$ when i different from j.

Note that since $\mathcal{T}_{xy} = \mathcal{T}_{yx}$, these symmetrical shear stresses always act simultaneously at the same point $(x,\, y,\, z)$, but on ortogonal planes through that point, **but I do not use points, but small cells** (Fry-Miranda) **Sequential** simplified view: Vector displacement:

$$\vec{d} = u\hat{\boldsymbol{\imath}} + v\hat{\boldsymbol{\jmath}} + w\hat{\boldsymbol{k}} = u_1\hat{\boldsymbol{\imath}} + u_2\hat{\boldsymbol{\jmath}} + u_3\hat{\boldsymbol{k}} = \vec{u}$$

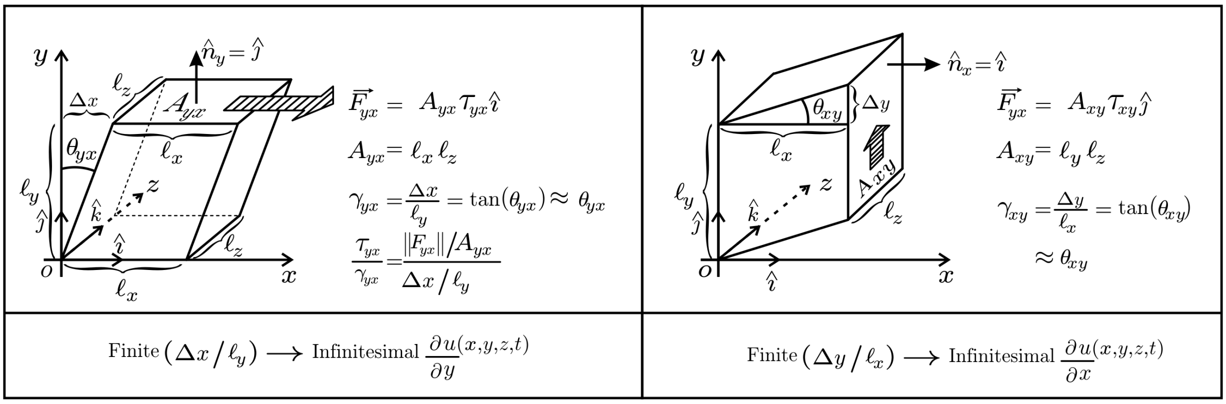


Figure 4. Infinitesimal strains under shear stresses.

2. Vibrating String Model Following Fry's Suggestion.

Under vector tension per unit cross -sectional area, $\vec{F}_+$ tangentially directed, and $\left\|\vec{F}_+\right\| = \text{constant} = T_0$.

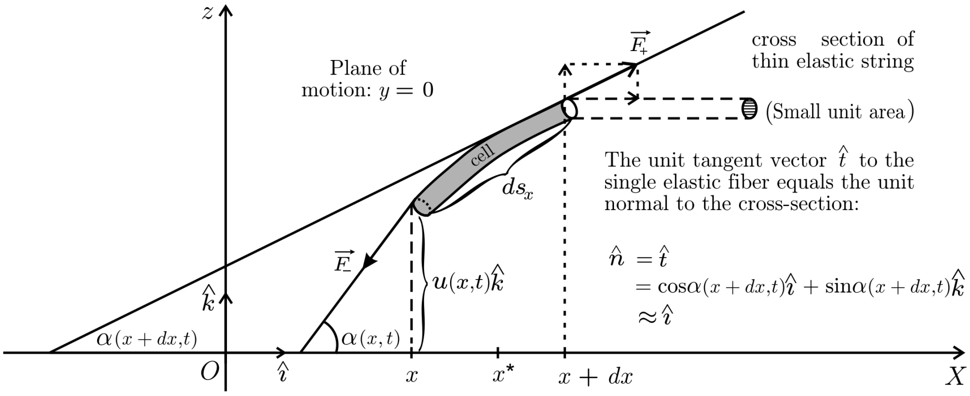


Figure 5. Infinitesimal elastic string slab.

$$\vec{F}_+ = T_0 \sin\alpha(x+dx,\,t)\hat{\boldsymbol{k}} + T_0\cos\alpha(x+dx,\,t)\hat{\boldsymbol{\imath}}; \qquad \tan\alpha(x+dx,\,t) = \frac{\partial u}{\partial x}(x+dx,\,t).$$

Since $\hat{t} \approx \hat{\boldsymbol{\imath}}$, then $(T_0 \sin\alpha(x+dx,t)\hat{\boldsymbol{k}})$ acts as s Shear Stress directed in the OZ direction, on a surface with normal $\hat{\boldsymbol{n}} \approx \hat{\boldsymbol{\imath}}$, or as $(\mathcal{T}_{xz}\hat{\boldsymbol{k}})$.

Approximating (for small amplitude vibrations):

$$\sin\alpha \approx \tan\alpha = \frac{\partial u}{\partial x}, \qquad \cos\alpha \approx 1$$

$$ds_x = \sqrt{1+\left(\tfrac{\partial u}{\partial x}\right)^2}dx \approx dx, \qquad \vec{F}_+ \approx T_0\frac{\partial u}{\partial x}(x+dx,\, t)\hat{\boldsymbol{k}}$$

then the **resulting upwards force** upon mass of cell element $dm = \rho(x,\, t)ds_x = \rho_0(x)dx$:

$$\hat{\boldsymbol{k}}T_0\left[\sin\alpha(x+dx,\, t) - \sin\alpha(x,\, t)\right] \approx T_0\left[\frac{\partial u}{\partial x}(x+dx,\, t) - \frac{\partial u}{\partial x}(x,\, t)\right]\hat{\boldsymbol{k}}; \qquad \text{let } dx = x_i - x_{i-1},$$

$$\approx T_0 h\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)(x^*,\, t)\hat{\boldsymbol{k}}, \qquad \text{with} \quad h = x_i - x_{i-1}, \quad x^* = \tfrac{1}{2}(x_{i-1}+x_i) = \text{cell center abscissa}$$

Next, apply Newton's Law for momentum, approximate $\rho_0(x) = \rho_0$ constant, and use Fry-Miranda's model,

$$\rho_0 h\hat{\boldsymbol{k}}\frac{d^2}{dt^2}u(x_{i-1/2},\, t) = T_0 h\hat{\boldsymbol{k}}\,\mathrm{DIV}\,\mathrm{GRAD}\,u(x_{i-1/2},\, t), \quad \text{whith} \quad x_{i-1/2} = x^*$$

This is our semi semi-discrete formulation (together with J. Castillo) for the vibrating string, and the 2nd order equation reduces to a 1st order system.

This reduction is achieved, as usual, by setting $v = du/dt$ as an auxiliary unknown.

Vector Displacement:

- Index Notation:

  $$\vec{u} = u_1\hat{\boldsymbol{\imath}} + u_2\hat{\boldsymbol{\jmath}} + u_3\hat{\boldsymbol{k}} = u_3(x_1,\, t)\hat{\boldsymbol{k}},$$

  $u_1 = u_2 \equiv 0$, and no dependence on x_2, x_3.

- Letter Notation:

  $$\vec{d} = u\hat{\boldsymbol{\imath}} + v\hat{\boldsymbol{\jmath}} + w\hat{\boldsymbol{k}} = w(x,\, t)\hat{\boldsymbol{k}}, \qquad u = v \equiv 0,$$

  and no dependence upon y, z.

Since $u_1 = u_2 = 0$, **we have called** $u_3(x_1,\, t) \equiv u(x,\, t)$, so that $\vec{u}(x,\, t) = u(x,\, t)\hat{\boldsymbol{k}}$

We want a Linear Relation between the stress tensor $\mathcal{T}$ and the strain tensor ε such that $\mathcal{T}_{ij} = \mathcal{T}_{ji}$ when $i \neq j$. This will impose a symmetry condition upon ε, namely $\varepsilon_{ij} = \varepsilon_{ji}$.

The combined (not sequential) angular deformation due to the simultaneous action of $\mathcal{T}_{ij}$ and $\mathcal{T}_{ji}$ upon an infinitesimal rectangular cell **must be** then **set in a symmetrical manner**, and we define:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \text{(index notation)} \tag{7}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \approx \tfrac{1}{2}(\theta_{yx} + \theta_{xy}), \quad \text{as we already saw} \tag{8}$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{9}$$

$$\boxed{\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)} \tag{10}$$

Also,

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \qquad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}, \qquad \text{(Index Notation)}$$

and

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}.$$

$$\begin{aligned} \text{Trace of } \varepsilon = \mathrm{Tr}(\varepsilon) &= \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \text{Volumetric Strain} = \Delta\, Vol/Vol(\text{unstrained}) \\ &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \operatorname{div} \vec{d} \\ &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \operatorname{div} \vec{u} \end{aligned}$$

1D vibrating string example: $u_1 = u \equiv 0, \quad u_2 = v = 0, \quad u_3 = u(x,\, t)$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \equiv 0, \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \equiv 0, \qquad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = \frac{\partial}{\partial z} u(x,\, t) \equiv 0,$$

so that $\mathrm{Tr}(\varepsilon) = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \equiv 0$, and Hooke's Law reads $\tau = 2\mu\varepsilon$, but

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \equiv 0, \qquad \varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \equiv 0,$$

and

$$\varepsilon_{31} = \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) = \frac{1}{2}\frac{\partial u_3}{\partial x_1} = \frac{1}{2}\frac{\partial}{\partial x}u(x,\, t)$$

and from Hooke's simplified Law for $\mathcal{T}_{xz}$

$$\mathcal{T}_{zx} = \boxed{\mathcal{T}_{xz}} = 2\mu\varepsilon_{xz} = 2\mu\varepsilon_{zx} = \boxed{\mu\frac{\partial}{\partial x}u(x,\, t)} \quad \text{(Recall that I named } w = u_3 = u\text{)}$$

and we see that the constant T_0 must be Lamé's second parameter. Also $\frac{\partial u}{\partial t}\hat{\boldsymbol{k}} = \frac{\partial u_3}{\partial t}\hat{\boldsymbol{k}}$. Newton's Law:

$$\rho\frac{\partial^2 u_3}{\partial t^2}\hat{\boldsymbol{k}} = \rho\frac{\partial}{\partial t}\left(\frac{\partial u_3}{\partial t}\right)\hat{\boldsymbol{k}} = \frac{\partial \mathcal{T}_{zx}}{\partial x}\hat{\boldsymbol{k}} = \frac{\partial \mathcal{T}_{xz}}{\partial x}\hat{\boldsymbol{k}} = \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)\hat{\boldsymbol{k}} = \frac{\partial}{\partial x}\left(T_0\frac{\partial u}{\partial x}\right)\hat{\boldsymbol{k}}$$

3. Combined Time – Symplectic / Space CG – Mimetic Difference Operators for the 1D IBVP for the Wave Equation $\frac{\partial^2 u}{\partial t^2} = c^2$ div grad u in $[0,\, \pi] \times [0,\, \infty)$.

The difference scheme uses the following grids:

$$\left.\begin{array}{lll} \text{2nd order : Position Newton } - \text{ Verlet} & / & D_x^2 G_x^2 \\ \text{4th order : Position Forest } - \text{ Ruth} & / & D_x^4 G_x^4 \end{array}\right\} \begin{array}{l} \underline{\text{staggered spatial grid: } n = \text{number of cells}} \\ \left[x_0,\, x_{1/2},\, x_1,\, x_{3/2}, \ldots, x_{n-1},\, x_{n-1/2}, x_n\right] \end{array}$$

- **Mesh Size:**

$$\Delta x = x_i - x_{i-1} = h, \qquad i = 1,\, 2,\, \ldots n, \quad \text{with} x_0 = 0, \ \text{and} \ x_n = \pi$$

- **Space Staggering:**

$$\text{Cell Centers } \ x_{i-1/2} = (x_{i-1} + x_i)/2 = (x_{i-1} + h/2), \qquad x_n = n(\Delta x) = nh.$$

- **Time Step:**

$$\Delta t = t_{k+1} - t_k, \qquad k = 0,\, 1,\, 2, \qquad t_k = k(\Delta t)$$

- **Time Staggering:**

$$\text{Introduce, for Newton } - \text{ Verlet, } t_{k+1/2} = (t_k + t_{k+1})/2, \quad k = 0,\, 1,\, 2, \ldots$$

and, for 4-th order symplectic Forest–Ruth, with $\theta = \frac{1}{2-\sqrt[3]{2}} = 1{,}351207\ldots$, **introduce** the following symmetrical intermediate steps, **starting from the known values** u_{ik} and $v_{ik} = \dot{u}_{ik}$ $(u_{ik} = u(x_{i-1/2},\, t_k))$ at time t_k, $1 \leq i \leq n$ $\ u_{ok} = u(x_0, t_k) = 0$, and $u(x_n,\, t_k) = 0$ for all $k = 0,\, 1,\, 2,\, \ldots$ but $\ u_{nk} \neq u(x_n,\, t_k)$.

$$\begin{array}{lcll}
(u_i)^{*}_{k} & = & u_{ik} + \theta(\Delta t/2)v_{ik} & \longleftrightarrow \quad \theta(\Delta t/2) \\[2ex]
(v_i)^{**}_{k} & = & v_{ik} + \theta(\Delta t)c^2 D^4_x G^4_x (u_i)^{*}_{k} & \longleftrightarrow \quad 2\theta(\Delta t/2) \\[2ex]
(u_i)^{**}_{k} & = & (u_i)^{*}_{k} + (1-\theta)(\Delta t/2)(v_i)^{**}_{k} & \longleftrightarrow \quad (1-\theta)(\Delta t/2) \\[2ex]
(v_i)^{***}_{k} & = & (v_i)^{**}_{k} + (1-2\theta)(\Delta t)c^2 D^4_x G^4_x (u_i)^{**}_{k} & \longleftrightarrow \quad 2(1-2\theta)(\Delta t/2) \\[2ex]
(u_i)^{***}_{k} & = & (u_i)^{**}_{k} + (1-\theta)(\Delta t/2)(v_i)^{***}_{k} & \longleftrightarrow \quad (1-\theta)(\Delta t/2) \\[2ex]
(v_i)_{k+1} & = & (v_i)^{***}_{k} + \theta(\Delta t)c^2 D^4_x G^4_x (u_i)^{***}_{k} & \longleftrightarrow \quad 2\theta(\Delta t/2) \\[2ex]
(u_i)_{k+1} & = & (u_i)^{***}_{k} + \theta(\Delta t/2)(v_i)_{k+1} & \longleftrightarrow \quad \theta(\Delta t/2)
\end{array}$$

Observe that due to the symmetry,

$$(\theta + 2\theta + 1 - \theta) + (4 - 4\theta) + (1 - \theta + 2\theta + \theta) = 4$$

independent of θ, which appears as an auxiliary numerical artifact, but necessary due to the theory of exponential integrators.

Even though symplectic algorithms possess global stability, it is convenient to keep (Δt) restricted by a **Courant − Friedrichs − Lewy condition:** namely, $(c\Delta t)/(\Delta x) \leq 1$, and numerical runs were carried out with $(c\Delta t)/(\Delta x) = 1/2$, or $(\Delta t) = (1/2c)\Delta x$.

The minimal spatial mesh size, $\Delta x = h$, **must be chosen so that the number of grid points per wave length** (ppw) **Samples Adequately** the initial data discretizing the continuous initial displacement and velocity for the manufactured exact solution of

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

in the space − time domain $[0,\, \pi] \times [0,\, +\infty)$, under homogeneous Dirichlet Boundary conditions, $u(0,\, t) = u(\pi,\, t) \equiv 0$ for all $t \geq 0$, namely: $u(x,\, t) = \sin x \cos{(ct)}$, and discrete $u(x_0,\, t_k) = u(0,\, t_k) \equiv 0$, $u(x_5,\, t_k) = u(\pi,\, t_k) \equiv 0$.

Our algorithm (hand made), exemplified only with second order operators D^2_x and G^2_x, and only 5 cells for simplicity, is a discrete scheme, with uniform high–order of accuracy using mimetic Castillo–Grone operators for space, and time–symplectic, jointly applied to the IBVP for the vibrating elastic string with normalized length π.

Continuous Formulation **Discrete in:**

$$\begin{cases} \dfrac{\partial u}{\partial t} = v & 0 < x < \pi = 180^\circ, \\[2ex] \dfrac{\partial v}{\partial t} = c^2 \operatorname{div} \operatorname{grad} u, & t > 0 \end{cases}$$

Position Newton – Verlet :$u_{ik} = u_i(k\Delta t), \quad v_{ik} = v_i(k\Delta t)$

Since x, u are dimensionless variables "c" has dimension (time)$^{-1}$, and so has "v".

Initial Conditions

$$\left.\begin{aligned} u(x,\,0) &= \sin x \\ \frac{\partial}{\partial t}u(x,\,0) &= 0 \end{aligned}\right\} 0 \le x \le \pi$$

Boundary Conditions

$u(0,\,t) = u(\pi,\,t) = 0, \quad t \ge 0$

Discrete

$u_{ik} = u(x_{i-1/2},\,t_k)$ only for $1 \le i \le n = 5$, that is, $u(x_{1/2},\,t_k)$, $u(x_{3/2},\,t_k)$, $u(x_{5/2},\,t_k)$ and $u(x_{7/2},\,t_k)$, $u(x_{9/2},\,t_k)$,

$$\left\{\begin{aligned} u_{0k} &= u(x_0,\,t_k) = u(0,\,t_k) \\ u_{6k} &= u(x_5,\,t_k) = u(\pi,\,t_k) \end{aligned}\right\} \begin{array}{l}\text{Exceptional} \\ \text{Boundary Nodes}\end{array}$$

Exact Sol^N: $u = \sin x \cos{(ct)}$, so that Initial Conditions to be discretized are:

$$u(x,\,0) = \sin x, \quad 0 \le x \le \pi$$

$$\frac{\partial}{\partial t}u(x,\,0) = 0, \quad 0 \le x \le \pi$$

The stencil for D has to avoid "overlapping" of **Block** submatrices, so $n > 4$.

A "minimal" stencil in the case of $D_x^2 G_x^2$, that also samples adequately the function $(\sin x)$ with 7 ordinates, namely, with $n = 5$, $\Delta x = 180^\circ/5 = 36^\circ$, $\frac{\Delta x}{2} = 18^\circ$,

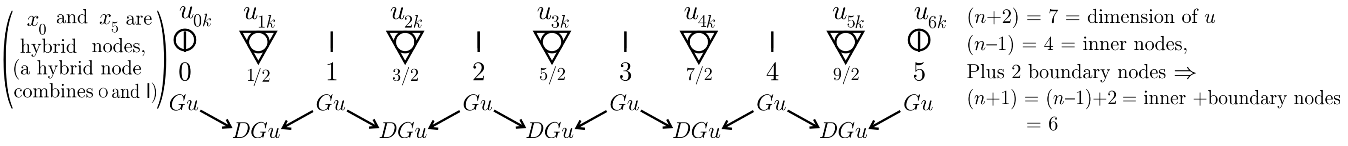
$$u_{00} = \sin{(0^\circ)}, \qquad u_{10} = u(x_{1/2},\,0) = \sin(18^\circ), \qquad u(x_{3/2},\,0) = \sin 54^\circ = u_{20},$$

$$u_{30} = u(x_{5/2},\,0) = \sin{(90^\circ)}, \quad \text{etc}, \qquad u_{50} = u(x_{9/2},\,0) = \sin{(162^\circ)} \ne \text{ from } u(x_5,\,0) = \sin{(180^\circ)} = u_{60}$$

will be now exhibited, but reverting to the fractional indexing, which is the standard one while using Castillo-Grone operators D and G, and temporarily avoided in order to describe easily Newton - Verlet, Forest-Ruth, that is, for the description of G, D and DGu, in the case $n = 5 =$ number of cells, the second order matrices G_x^2, D_x^2 will be $[n+1] \times [n+2] = 6 \times 7$, and $[n \times (n+1)] = 5 \times 6$ so that $D_x^2 G_x^2$ will be $[n \times (n+2)] = 5 \times 7$ acting upon $u = [(n+2) \times 1] = [u_0,\, u_{1/2},\, u_{3/2},\, u_{5/2},\, u_{7/2},\, u_{9/2}, u_6]^T = [7 \times 1]$ to yield

$$D^2 G^2 u = [5 \times 7] \times [7 \times 1] = [5 \times 1] = \left[DGu(x_{1/2}),\, DGu(x_{3/2}),\, DGu(x_{5/2}),\, DGu(x_{7/2}),\, DGu(x_{9/2})\right]^T.$$

The numerical values of u, Gu, DGu **are bound to** different staggered grid points **according to Fig. 4.2. from Castillo-Miranda's Book [1]**:



Referring again to Fig. 4.2, we see that for $n = 5$ cell center nodes, Gu values are bound to $(n+1) = 6$ nodes | (all nodes)(inner + boundary nodes).

u values are bound to $(n+2) = 7$ (cell center nodes + boundary nodes) = ◯. DGu values are bound to $n = 5$ cell center nodes ▽ only.

Stencils for 2nd order Castillo-Grone G and D

$$\overset{[\,v\,]}{\begin{bmatrix}v_0\\v_1\\v_2\\v_3\\v_4\\v_5\end{bmatrix}} \equiv \underbrace{\overset{G[\,u\,]}{\begin{bmatrix}Gu|_0\\Gu|_1\\Gu|_2\\Gu|_3\\Gu|_4\\Gu|_5\end{bmatrix}}}_{[6\times1]} = \left(\frac{1}{h}\right) \underbrace{\overset{\longleftarrow\; G \;\longrightarrow}{\overset{\frac{3}{2}k=3}{k=2\left\{\left[\begin{array}{ccccccc} -\frac{8}{3} & 3 & -\frac{1}{3} & 0 & 0 & 0 & 0\\ \text{upper} & \text{left} & \text{block} & & & & \\ 0 & -1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{3} & -3 & \frac{8}{3}\end{array}\right]\right.}}}_{\text{3 columns} \qquad\qquad \text{3 columns}\;\;[6\times7]} \underbrace{\overset{[\,u\,]}{\begin{bmatrix}u_0\\u_{1/2}\\u_{3/2}\\u_{5/2}\\u_{7/2}\\u_{9/2}\\u_5\end{bmatrix}}}_{[7\times1]}$$

$$\left(\frac{9u_{1/2} - u_{3/2} - 8u_0}{3h} = u_0' + \frac{1}{8}h^2 u_0'''\right)$$

(Observe the upper left and the lower right boundary blocks [2×3], they Do Not "Overlap" along the column indexing direction: 6+1=7, 5=7−2.)

Note: n minimal $= 3k - 1 = 6 - 1 = 5$ (See formula (J.2) in **Appendix J**, reference [1])

$$D[\,v\,] = DG[\,u\,] = L[\,u\,]$$

$$\underbrace{\begin{bmatrix}Dv|_{1/2}\\Dv|_{3/2}\\Dv|_{5/2}\\Dv|_{7/2}\\Dv|_{9/2}\end{bmatrix}}_{[5\times1]} = \left(\frac{1}{h}\right)\underbrace{\begin{bmatrix}-1 & 1 & 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & 0 & 0\\ 0 & 0 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0 & -1 & 1\end{bmatrix}}_{[5\times6]} \underbrace{\begin{bmatrix}v_0\\v_1\\v_2\\v_3\\v_4\\v_5\end{bmatrix}}_{[6\times1]} = \underbrace{D}_{[5\times6]}\underbrace{G}_{[6\times7]} \underbrace{\overset{\text{Fractional index}}{\begin{bmatrix}u_0\\u_{1/2}\\u_{3/2}\\u_{5/2}\\u_{7/2}\\u_{9/2}\\u_5\end{bmatrix}}}_{[7\times1]} \left.\begin{array}{l}\text{Integer index}\\ = u_{0k}\\ = u_{1k}\\ = u_{2k}\\ = u_{3k}\\ = u_{4k}\\ = u_{5k}\\ = u_{6k}\end{array}\right\}\text{Old Positions}$$

After multiplying D and G get for L:

$$\underbrace{\begin{bmatrix} Lu|_{1/2} \\ Lu|_{3/2} \\ Lu|_{5/2} \\ Lu|_{7/2} \\ Lu|_{9/2} \end{bmatrix}}_{[5\times 1]} = \left(\frac{1}{h^2}\right) \underbrace{\begin{array}{c} \\ k=2\left\{ \begin{array}{c} \\ \\ \end{array}\right. \\ \\ \\ \\ \end{array}\begin{array}{ccc|c|ccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \frac{8}{3} & -4 & \frac{4}{3} & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{3} & -4 & \frac{8}{3} \end{array}}_{[5\times 7]} \underbrace{\begin{bmatrix} u_0 \\ u_{1/2} \\ u_{3/2} \\ u_{5/2} \\ u_{7/2} \\ u_{9/2} \\ u_5 \end{bmatrix}}_{[7\times 1]} \begin{array}{l} = u_{0k} \\ = u_{1k} \\ = u_{2k} \\ = u_{3k} \\ = u_{4k} \\ = u_{5k} \\ = u_{6k} \end{array} = \left(\frac{1}{h^2}\right) \underbrace{S}_{[5\times 7]} [\, u \,]$$

The New Positions $u_{i\,k+1}$, are obtained by Newton Verlet.

Since x is a dimensionales variable, so is $h = \Delta x = x_i - x_{i-1}$.

Let us apply second order time symplectic Position Newton – Verlet (2nd order)

$$\begin{array}{ll} \text{Aux. Positions} & \left\{ \begin{array}{llll} u_{ik+1/2} = & \overset{\text{Old Positions}}{\widetilde{u_{ik}}} + (\Delta t/2) + \overset{\text{Old Velocities}}{\widetilde{v_{ik}}}, & \text{and} & u_{0k+1/2} = u(x_5, t_{k+1/2}) = 0 \\ v_{ik+1} = & v_{ik} + (\Delta t)c^2 \frac{1}{h^2}(Su)_{ik+1/2}, & \text{for} & i = 1,\, 2,\, \dots,\, n = 5 \\ u_{ik+1} = & u_{ik+1/2} + (\Delta t/2) v_{ik+1}, & \text{and} & u_{0k+1} = u(x_5, t_{k+1}) = u_{6k+1} = 0 \end{array} \right. \\ \text{New Velocities} & \\ \text{New Positions} & \end{array}$$

(**Observe:** There is No Auxiliary velocity at $k + 1/2$ because of "Jump"then).

- **First Run**, Starts with initial data u_{i0}, v_{i0}, $x_0 = 0°$, $x_5 = 180°$

  - Seven (5+2) values: $\left\{ \begin{array}{llll} u_{00} = 0 = \sin 0°, & u_{10} = \sin(18°), & u_{20} = \sin(54°), & u_{30} = \sin(90°), \\ u_{40} = \sin 126°, & u_{50} = \sin(162°), & u_{60} = u(x_5,\, 0) = \sin 180° & \end{array} \right.$

    and zero initial velocities, so that:

  - $k = 0$. $\left\{ \begin{array}{lllllll} u_{00} = 0, & u_{10} = \sin(18°), & u_{20} = \sin(54°), & u_{30} = \sin(90°), & u_{40} = \sin 126°, & & \\ u_{50} = \sin(162), & u_{60} = 0 & & & & & \\ v_{00} = 0, & v_{10} = 0, & v_{20} = 0, & v_{30} = 0, & v_{40} = 0, & v_{50} = 0, & v(x_5,\, 0) = 0 \end{array} \right.$

    auxiliary positions at $t_{k+1/2} = t_{1/2}$:

- k+1/2=1/2.
  k+1 = 1

$$\begin{cases} u_{01/2} = 0, \quad u_{11/2} = u_{10} + (\Delta t/2)v_{10}, \\ u_{21/2} = u_{20} + (\Delta t/2)v_{20}, \ldots, u_{51/2} = u_{50} + (\Delta t/2)v_{50}, \quad u(x_5, t_{1/2}) = 0. \\ \text{NewVelocities} \\ v_{01} = 0, \;\; v_{11} = v_{10} + (\Delta t)\frac{c^2}{h^2}\left[\frac{8}{3}u_01/2 - 4u_{11/2} + \frac{4}{3}u_{21/2}\right]\cdots, v(x_5, t_1) = 0. \\ \text{NewPositions} \\ u_{01} = 0, \;\; u_{11} = u_{11/2} + (\Delta t/2)v_{11}, \ldots, u(\underbrace{x_5}_{u_{61}}, t_1) = 0 \end{cases}$$

- **Second Run**, starts with u_{i1}, v_{i1}.

- k+1/2=3/2.
  k+1 = 2

$$\begin{cases} u_{03/2} = 0, \;\; u_{13/2} = u_{11} + (\Delta t/2)v_{11}, \;\; u_{23/2} = u_{21} + (\Delta t/2)v_{21}, \ldots, u(x_5, t_{3/2}) = 0. \\ v_{02} = 0, \;\; v_{12} = v_{11} + (\Delta t)\frac{c^2}{h^2}\left[\frac{8}{3}u_{03/2} - 4u_1 v_2 + \frac{4}{3}u_{23/2}\right] \quad \cdots\cdots, \quad v(x_5, t_{3/2}) = 0. \\ u_{02} = 0, \;\; u_{12} = u_{13/2} + (\Delta t/2)v_{12}, \quad \ldots\ldots, \qquad u(\underbrace{x_5}_{u_{62}}, t_1) = 0. \end{cases}$$

This time–space algorithm was tested with a pocket Casio calculator, and worked perfectly with $(c\Delta t/\Delta x) = 1$ and some physically reasonable value of c.

To get fourth–order time–space, use D^4_x, G^4_x and a larger number of cells because the "minimal" stencils are "bigger" in the number of non-zero entries for their respective rows.

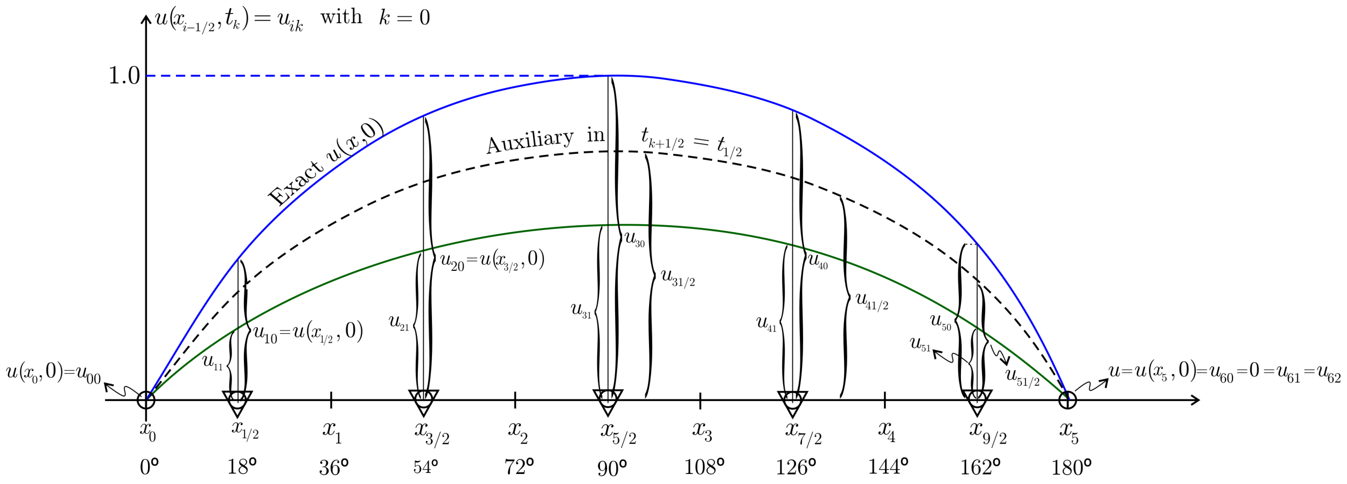


Figure 6. .Exact initial string displacement and computed u_{i1}, and auxiliary $u_{i1/2}$ with 2nd order position Newton–Verlet

To simplify, use $c\Delta t/\Delta x = 1$, so that

$$c^2 \Delta t \left(\frac{1}{\Delta x}\right)^2 = \left(\frac{c}{\Delta x}\right)\left(\frac{c\Delta t}{\Delta x}\right) = \frac{c}{\Delta x}.$$

For a minimal stencil, $n = 5$, $\Delta x = \pi/5$, and for numerical expediency, choose $c = 100(\pi/5)$, so that $c/\Delta x = 100$ and then

$$\Delta t = \frac{\Delta x}{c} = \frac{1}{100} = 0.01, \qquad \frac{\Delta t}{2} = 0.005; \qquad \Delta x = 0.628, \qquad (\Delta x)^2 = 0.39$$

Old Initial Pos^N	u_{00}	u_{10}	u_{20}	u_{30}	u_{40}	u_{50}	u_{60}
	0	0.30902	0.80902	1.0	0.80902	0.30902	0
Old Initial Vel	$v_{00} = 0$	$v_{10} = 0$	$v_{20} = 0$	$v_{30} = 0$	$v_{40} = 0$	$v_{50} = 0$	$v_{60} = 0$
Aux Positions	$u_{01/2}$	$u_{11/2}$	$u_{21/2}$	$u_{31/2}$	$u_{41/2}$	$u_{51/2}$	$u_{61/2}$
	0	0.30902+0	0.80902+0	1.0+0	0.80902+0	0.30902+0	0.
New Velocs	v_{01}	v_{11} $\cdots$	v_{21} $\cdots$	v_{31} $\cdots$	v_{41} $\cdots$	v_{51} $\cdots$	v_{61}=0
	0	-15.739^*					
New Positions	u_{01}	u_{11}					
	0	0.23002^{**}					

Note:

$^* = 0 + 100\left[\frac{8}{3} \times 0 - 4 \times 0.30902 + \frac{4}{3} \times 0{,}80902\right] = 100[-0.15739 = -15.739$

$^{**} = 0.30902 + 0 + 0.005(-15.739) = 0.30902 - 0.079 = 0.23002$

Exact New Position is (Grid too Coarse,

$$u(x_{1/2},\, \Delta t) = u(18^\circ,\, \Delta t) = \sin(18^\circ)\cos\overbrace{(c\Delta t)}^{36^\circ} = 0.30902 \times 080902 = 0.25, \quad \text{error} = 0.02)$$

This exhibits how the second order algorithm (Position Newton–Verlet, and D_x^2, G_x^2) advances from k to $(k+1)$, having a single "kicking term" acting at $(k+1/2)$ on the only Auxiliary Position $u_{iK+1/2}$.

This is to be contrasted with fourth order Forest – Ruth, combined this time with D_x^4, G_x^4, where **three** "kicking terms" are needed between k and $(k+1)$, and the added computational cost is the price to be paid for the enhanced combined higher order accuracy and the symplectic feature of the Forest – Ruth time – stepping, which brings in time reversibility and long term stability.

4. Stress Tensor Components.

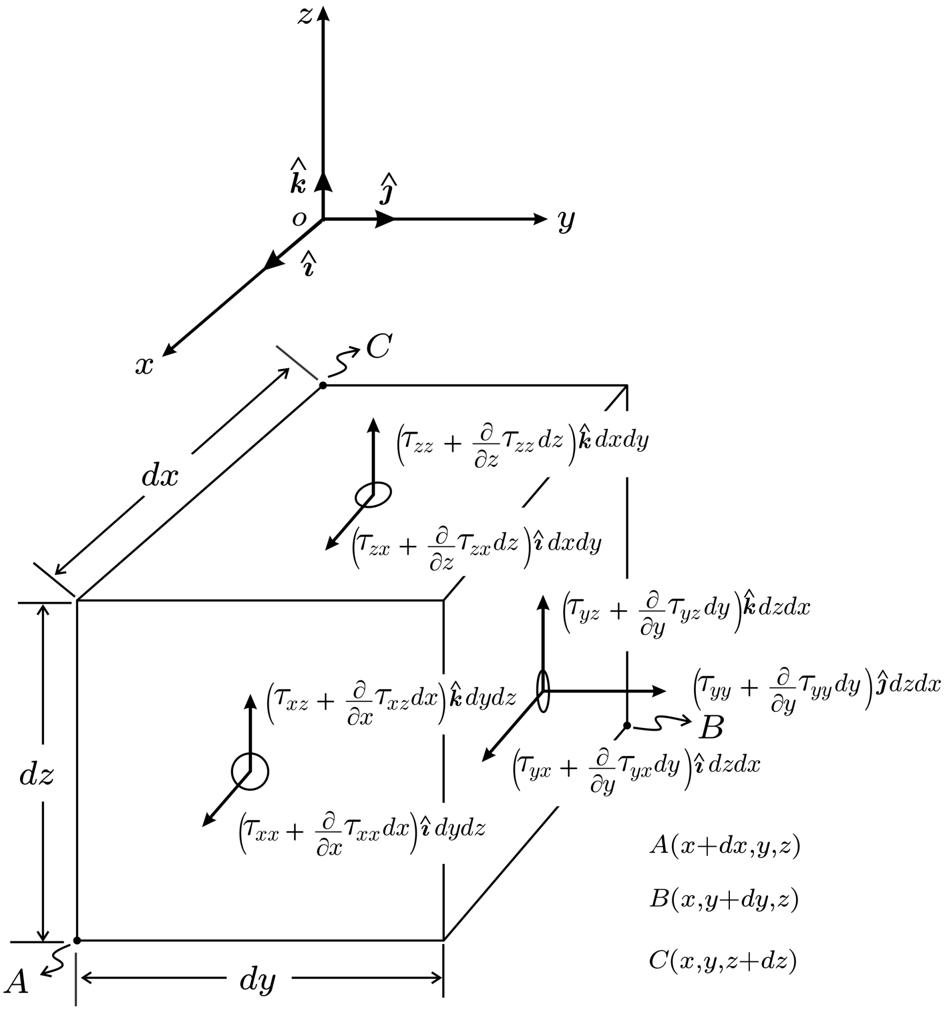


Figure 7. Stresses acting on the visible faces of the infinitesimal $(dx\,dy\,dz)$ parallelepiped.

There are **no** internal stresses $\mathcal{T}_{ij}$ if there are **no internal strains** ε_{ij}, **and conversely**, there are **no** internal strains if there are **no external** stresses $\vec{t}_{\text{ext}}$.

Causal chain:

$$\vec{t}_{\text{ext}} \longrightarrow \text{internal strain } \varepsilon_{ij} \xrightarrow{\quad Hooke \quad} \text{int stress}\mathcal{T}_{ij}$$

Detailed view of cross – section at x, **under rotation**, under the simplified rectilinear deformation seen in page 2 for $x > 0$: rotated cross – sectional area A is greater than A_x:

$$A_x = A\langle \hat{\boldsymbol{n}}, \hat{\boldsymbol{\imath}} \rangle = A\cos\alpha$$

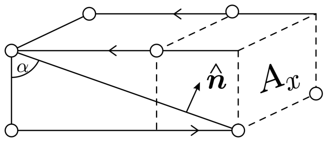


Figure 8. Rotated cross-section at x, detailed.

Newton's Momentum equation

For displacement vector $\vec{d} = u_1\hat{\boldsymbol{\imath}} + u_2\hat{\boldsymbol{\jmath}} + u_3\hat{\boldsymbol{k}}$ **along $\hat{\boldsymbol{k}}$**, applied to mass $dm = \rho dxdydz$

$$\left\langle \hat{\boldsymbol{k}},\, \rho\frac{\partial^2\vec{d}}{\partial t^2}dxdydz \right\rangle = \rho dxdydz\frac{\partial^2 u_3}{\partial t^2} = dxdydz\left[\frac{\partial \mathcal{T}_{xz}}{\partial x} + \frac{\partial \mathcal{T}_{yz}}{\partial y} + \frac{\partial \mathcal{T}_{zz}}{\partial z}\right] + \rho dxdydz\langle \hat{\boldsymbol{k}}, \vec{b}\rangle$$

or, in case body forces vanish, $\vec{b} = \vec{0}$: (thus neglecting, $\vec{b} = -g\hat{\boldsymbol{k}}$)

$$\rho\frac{\partial^2 u_3}{\partial t^2} = \frac{\partial \mathcal{T}_{xz}}{\partial x} + \frac{\partial \mathcal{T}_{yz}}{\partial y} + \frac{\partial \mathcal{T}_{zz}}{\partial z}$$

Equation Along $\hat{\boldsymbol{\imath}}$: (useful for 1D Acoustic "elastic" wave equation)

$$\rho\frac{\partial^2 u_1}{\partial t^2} = \frac{\partial \mathcal{T}_{xx}}{\partial x} + \frac{\partial \mathcal{T}_{yx}}{\partial y} + \frac{\partial \mathcal{T}_{zx}}{\partial z}, \qquad (u_2 = u_3 \equiv 0, \qquad \mathcal{T}_{yx} = \mathcal{T}_{zx} \equiv 0)$$

1D Vibrating string: (**along** $\hat{\boldsymbol{k}}$) $u_1 = u_2 \equiv 0, \qquad \mathcal{T}_{yz} = \mathcal{T}_{zz} \equiv 0, \qquad \mathcal{T}_{xz} = T_0 \sin\alpha \approx T_0\dfrac{\partial u_3}{\partial x}$

$$\rho\frac{\partial^2 u_3}{\partial t^2} = \frac{\partial}{\partial x}\left(T_0\frac{\partial u_3}{\partial x}\right) = T_0\frac{\partial^2 u_3}{\partial x^2}.$$

Think of the string as a single elastic fiber with cross sectional rectangular area A with unit normal $\hat{\boldsymbol{n}}$: Since $u_1 = u_2 = 0$, rename $u_3 = u$

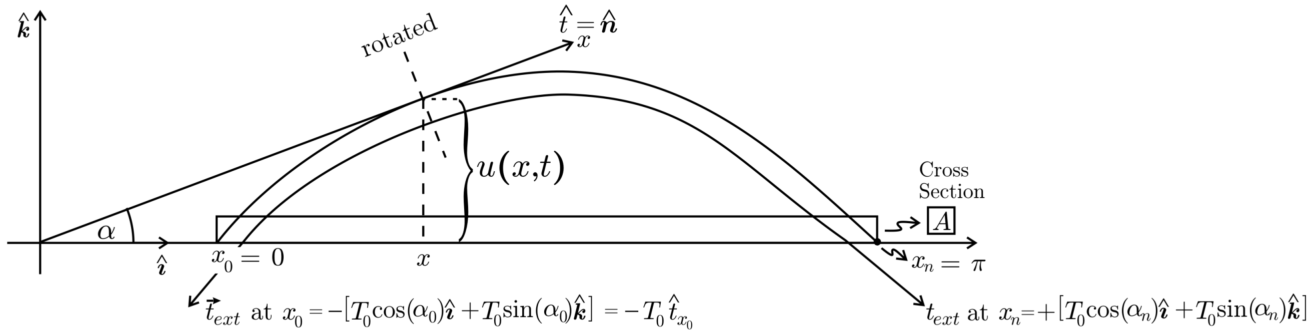


Figure 9. Rotated cross-section at x.

The string cross – section that was orthogonal to $\hat{\boldsymbol{\imath}}$ at abscissa x, after displacement $\vec{d} \approx u(x,\, t)\hat{\boldsymbol{k}}$, **becomes rotated**, and is now orthogonal to that tangent vector at $(x,\, u(x,\, t))$, that is to $(\cos\alpha\hat{\boldsymbol{\imath}} + \sin\alpha\hat{\boldsymbol{k}})$.

$$\mathcal{T}_{xx}\hat{\boldsymbol{\imath}} = T_0\cos\alpha\hat{\boldsymbol{\imath}}/A_x = T_0\hat{\boldsymbol{\imath}} \qquad \Rightarrow \qquad \mathcal{T}_{xx} = T_0$$

5. 1D - Acoustic Wave Equation under "Elastic" Presentation.

Assume a one dimensional displacement a long the x – axis, so that

$$u_2 = u_3 \equiv 0 \qquad \Rightarrow \qquad \varepsilon_{22} = \varepsilon_{33} \equiv 0 \qquad \Rightarrow \qquad \mathrm{Tr}(\varepsilon) = \varepsilon_{11} = \frac{\partial u_1}{\partial x_1}.$$

There is no y or z dependence, so that, writing $x_1 = x$, $u_1 = u$, we find:

$$\mathrm{Tr}(\varepsilon) = \frac{\partial u}{\partial x}(x,\, t).$$

Also, no shear stresses are considered, and the model assumes $\mathcal{T}_{yx} = \mathcal{T}_{zx} \equiv 0$, and write $\mathcal{T}_{11} = \mathcal{T}_{xx} = \mathcal{T}$, so that Newton's law along the x – axis simplifies to:

$$\rho \frac{\partial^2 u}{\partial t^2} = \rho \frac{\partial v}{\partial t} = \frac{\partial \mathcal{T}}{\partial x} \qquad \text{with} \quad v \equiv \frac{\partial u}{\partial t}, \qquad \text{or} \qquad \frac{\partial v}{\partial t} = \left(\frac{1}{\rho}\right) \frac{\partial \mathcal{T}}{\partial x}$$

To complete a first order system for v and $\mathcal{T}$, use Hooke's Law: $\mathcal{T}_{11} = \lambda \mathrm{Tr}(\varepsilon) + 2\mu\varepsilon_{11} = (\lambda + 2\mu)\varepsilon_{11}$, and since $\varepsilon_{11} = \frac{\partial u}{\partial x}$, get: $\mathcal{T} = (\lambda + 2\mu)\frac{\partial u}{\partial x}$. Renaming $(\lambda + 2\mu) = \mu^*$ we obtain a complete first order system of hyperbolic PDE's, to be solved in the space – time domain $\Omega \times [0,\, \infty] = [0,\, 1] \times [0,\, \infty]$

$$\left\{ \begin{array}{l} \dfrac{\partial v}{\partial t} = \left(\dfrac{1}{\rho}\right) \dfrac{\partial \mathcal{T}}{\partial x} \\[2ex] \dfrac{\partial \mathcal{T}}{\partial t} = \mu^* \dfrac{\partial}{\partial t}\left(\dfrac{\partial u}{\partial x}\right) = \mu^* \dfrac{\partial v}{\partial x} \end{array} \right\} \begin{array}{l} 0 \le x \le 1 \\[2ex] 0 \le t \end{array}$$

This system is subjected to initial conditions and Homogeneous Boundary Conditions for $\mathcal{T}$:

- I.C. $\left\{ \begin{array}{l} v(x,\, 0) = v_0(x) \\[1ex] \mathcal{T}(x,\, 0) = \mathcal{T}_0(x) \end{array} \right\} \; 0 \le x \le 1$
- B.C. $\left\{ \begin{array}{l} \mathcal{T}(0,\, t) = \mathcal{T}(1,\, t) \equiv 0 \\[1ex] \dfrac{\partial v}{\partial x}(0,\, t) = \dfrac{\partial v}{\partial x}(1,\, t) \equiv 0 \end{array} \right\} \; \text{for all} \;\; t \ge 0$

Both v and $\mathcal{T}$ must satisfy the 1-D wave equations with sound speed $c = \sqrt{\mu^*/\rho}$:

$$\rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{T}}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{T}}{\partial t}\right) = \frac{\partial}{\partial x}\left(\mu^* \frac{\partial v}{\partial x}\right) = \mu^* \frac{\partial^2 v}{\partial x^2}$$

and

$$\frac{\partial^2 \mathcal{T}}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{T}}{\partial t}\right) = \frac{\partial}{\partial t}\left(\mu^* \frac{\partial v}{\partial x}\right) = \mu^* \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) = \mu^* \frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial \mathcal{T}}{\partial x}\right) = \left(\frac{\mu^*}{\rho}\right) \frac{\partial^2 \mathcal{T}}{\partial x^2}.$$

The manufactured solution for the Initial Boundary Value Problem with $v_0(x) = 0$ and $\mathcal{T}_0(x) = \sin(\pi x)$ for $0 \le x \le 1$ is:

$$\mathcal{T}(x,\, t) = \sin(\pi x)\cos(\pi c t), \qquad v(x,\, t) = \left(\frac{c}{\mu^*}\right)\cos(\pi x)\sin(\pi c t).$$

In The 79$^{\text{th}}$ EAGE Conference & Exhibition 2017, held in Paris, France, Rojas, O. et al [2]. used staggered FD stencils, with a 2009 strategy with the following storage rules on a 1D-staggered grid with N cells, $(N+1)$ nodes $x_i = ih$, $0 \le i \le N$, $h = 1/N$, and N cell centers $x_{i+1/2} = (x_i + x_{i+1})/2$:

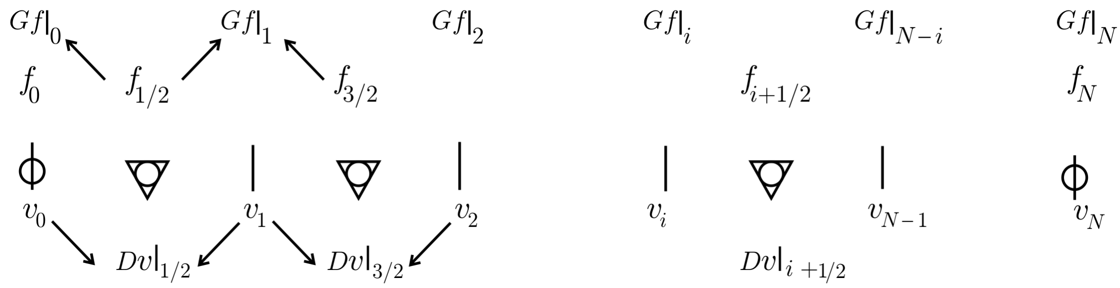
- **A)** The spatial numerical differentiations of the discrete velocities, to be multiplied by μ^*, are stored at cell centers, and computed from the discrete values of $v = v_i = v(x_i)$, and $\frac{\partial v}{\partial x}$ acts as a one – dimensional divergence of field v.
- **B)** On the other hand, the spatial numerical differentiations of the discrete stresses, to be multiplied by $(1/\rho)$, are stored at all grid nodes, and computed from the discrete values of $\mathcal{T} = \mathcal{T}_{1+1/2} = \mathcal{T}(x_{i+1/2})$, and $\frac{\partial \tau}{\partial x}$ acts as a one–dimensional gradient of field $\mathcal{T}$.

It is important, in order to change from Rojas, O., et al. finite differences to Castillo – Grone mimetic Operators DIV and GRAD, to compare, for Analogy purposes, the staggered – grid diagrams:

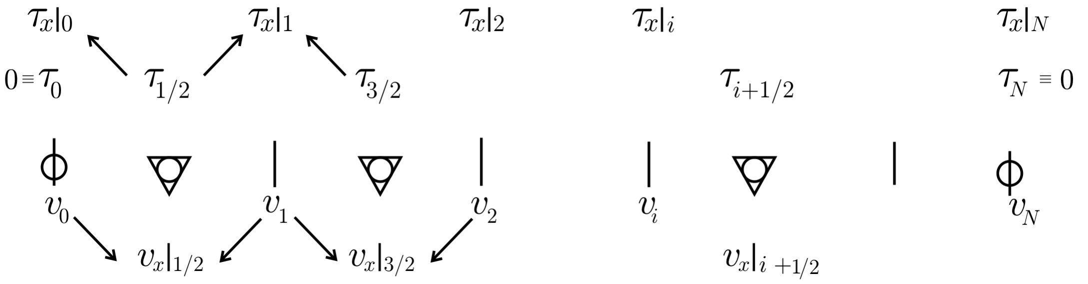
Same 1D-Staggered grid N Cells

Castillo - Grone Storage of fields

(Pag. 46, Fig. 4.2, Castillo–Miranda's Book [1])



Rojas, et. al. [2] Storage of fields, writing $\mathcal{T}_x = \dfrac{\partial \mathcal{T}}{\partial x}$, $v_x = \dfrac{\partial v}{\partial x}$



"Writing $\mathcal{T}_t = \frac{\partial \tau}{\partial t}$ and $v_t = \frac{\partial v}{\partial t}$, for time–stepping have to solve:

$$\mathcal{T}_{t\big|i+1/2} = \mu^* v_{x\big|i+1/2}, \qquad v_{t\big|i} = (1/\rho)\mathcal{T}_{x\big|i};$$

Rojas, O. et al. [2] used a Lax–Wendroff –Leap–Frog. Now have to use Newton – Verlet (2nd order in time) and then, Forest-Ruth (4th order).

This is done solving the semi-discrete system for $\mathcal{T}(t)$ and $\dot{\mathcal{T}}(t) \equiv \omega(t)$:

$$\begin{cases} \dot{\omega}(t) = c^2 \mathrm{Div}\,\mathrm{GRAD}\,\mathcal{T} \\ \dot{\mathcal{T}}(t) = \omega(t) \end{cases}$$

with Initial Conditions

$$\mathcal{T}(0) = \mathcal{T}_0(x_{i+1/2}), \qquad \omega(0) = \mu^* \frac{dv_0}{dx}(x_{i+1/2})$$

and Boundary Conditions for $\mathcal{T}$:

$$\mathcal{T}_0 = \mathcal{T}_N \equiv 0, \quad \text{for all } t \geq 0.$$

Castillo and Yasuda's treatment [3] for Laplace's equation with Robin Boundary Conditions, can be used to solve for $v(x_i, t)$. Robin includes Neumann Boundary condition as a particular case.

6. Elastic Waves in an Isotropic Homogeneous Medium.

Loosely speaking, Helmholtz classical decomposition theorem for smooth vector fields $\vec{u}$ in free space, says that such general fields can be written as a sum of $\vec{u}_{_L}$, a curl – free field, and $\vec{u}_{_T}$, a divergence–free field. Thus, we can write

$$\vec{u} = \vec{u}_{_L} + \vec{u}_{_T}, \quad \text{whith} \quad \vec{u}_{_L} = \mathrm{grad}\varphi = \nabla\varphi, \quad \text{and} \quad \vec{u}_{_T} = \mathrm{curl}\,\overrightarrow{\psi} = \nabla \times \psi,$$

for some smooth scalar and vector fields φ and $\overrightarrow{\psi}$ respectively. If this theorem is valid in some space H of smooth vector fields, and if we define two Helmholtz subspaces of H, namely, $H_{_L}$ for curl-free fields, and $H_{_T}$ for the divergence-free fields, then $H = H_{_L} + H_{_T}$.

Assuming Hooke's Law, the linearized momentum equation for the vector elastic displacement $\vec{u}$ is:

$$\rho\frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + \mu)\mathrm{grad}\,\mathrm{div}\vec{u} + \mu(\overrightarrow{\nabla})^2\vec{u},$$

with

$$\vec{\nabla}^2 \vec{u} = \mathrm{grad}\,\mathrm{div}\,\vec{u} - \mathrm{curl}\,\mathrm{curl}\vec{u} = \text{vector Laplacian of } \vec{u}.$$

Equivalently,

$$\rho\frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + 2\mu)\mathrm{grad}\,\mathrm{div}\vec{u} + \mu[-\mathrm{curl}\,\mathrm{curl}]\vec{u}$$

Two basic body waves.

First, let us set $\vec{u}_L = 0$ and $\vec{u} = \vec{u}_L = \mathrm{grad}\,\varphi$, $\mathrm{curl}\,\vec{u}_L = \mathrm{curl}\,\mathrm{grad}\varphi = 0$, then

$$\frac{\partial^2 \vec{u}_L}{\partial t^2} = \frac{(\lambda + 2\mu)}{\rho}\mathrm{grad}\,\mathrm{div}\,\vec{u}_L = \frac{(\lambda + 2\mu)}{\rho}\vec{\nabla}^2_L \vec{u}_L \equiv (c_L)^2 \vec{\nabla}^2_L \vec{u}_L, \quad \vec{u}_L \in H_L.$$

The longitudinal vector Laplacian $\vec{\nabla}^2_L = \mathrm{grad}\,\mathrm{div}$, is the restriction to H_L of $\vec{\nabla}^2$.

Secondly, let us set $\vec{u}_L = 0$ and $\vec{u} = \vec{u}_T = \mathrm{curl}\,\psi$, $\mathrm{div}\vec{u}_T = \nabla \cdot \nabla_x \vec{\varphi} = 0$, then

$$\frac{\partial^2 \vec{u}_T}{\partial t^2} = -\frac{\mu}{\rho}\mathrm{curl}\,\mathrm{curl}\vec{u}_T = \frac{\mu}{\rho}\vec{\nabla}^2_T \vec{u}_T \equiv (c_T)^2 \vec{\nabla}^2_T \vec{u}_T, \quad \vec{u}_T \in H_T.$$

The transversal vector Laplacian

$$\vec{\nabla}^2_T = [-\mathrm{curl}\,\mathrm{curl}],$$

is the restriction to H_T of $\vec{\nabla}^2$, and then, $\vec{\nabla}^2_L \vec{u}_L = \vec{\nabla}^2 \vec{u}_L$ when $\mathrm{curl}\,\vec{u}_L = 0$ and $\vec{\nabla}^2_T \vec{u}_T = \vec{\nabla}^2 \vec{u}_T$ when $\mathrm{div}\,\vec{u}_T = 0$.

Thus, we have found two types of elastic body waves, namely:

1) A longitudinal, or a wave of expansion–compression (the P - wave in seismology), with wave speed

$$c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho}} \tag{11}$$

2) A transversal or shear –wave (the S–wave in seismology), with wave speed

$$c_T = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{E}{2(1+\nu)\rho}}, \tag{12}$$

and obviously, c_L is always $> c_T$, and in fact, $c_L > \sqrt{\frac{4}{5}}c_T$.

7. Monochromatic Elastic Body and Surface Waves.

Given an angular frequency "ω", elastic vector displacements $\vec{u}(\vec{r},\, t)$ undergoing **time–harmonic vibrations**, can be expressed in the general form

$$\vec{u}(\vec{r},\, t) = \vec{u}_0(\vec{r})\, e^{-i\omega t}, \qquad \text{with} \quad \vec{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}.$$

The momentum equation **derived from Hooke's Law** can be rewritten in terms of $C_L = \sqrt{\frac{\lambda+2\mu}{\rho}}$ and $c_T = \sqrt{\frac{\mu}{\rho}}$:

$$\frac{\partial^2 \vec{u}}{\partial t^t} = (c_T)^2 \vec{\Delta}\, \vec{u} + \left(c_L^2 - c_T^2\right) \operatorname{grad}(\operatorname{div} \vec{u}). \tag{13}$$

$\left(\text{use } \mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \lambda + \mu = \frac{E}{2(1+\nu)(1-2\nu)}\right)$

From

$$\vec{u}(\vec{r},\, t) \equiv \vec{u}_0(\vec{r})\, e^{-i\omega t},$$

and after substituting this expression for $\vec{u}(\vec{r},\, t)$ in equation (13), we get "**Elastic Helmhotz**" **equation**:

$$(c_T)^2 \vec{\Delta} \vec{u}_0 + \left(c_L^2 - c_T^2\right) \operatorname{grad} \operatorname{div} \vec{u}_0 + \omega^2 \vec{u}_0 = 0.$$

Since

$$\vec{u}_0(\vec{r}) \equiv e^{i\omega t}\, \vec{u}(\vec{r},\, t), \qquad \Rightarrow \qquad \operatorname{curl} \vec{u}_0(\vec{r}) = e^{i\omega t} \operatorname{curl} \vec{u}(r,\, t) \qquad \text{and} \qquad \operatorname{div} u_0(\vec{r}) = e^{i\omega t} \operatorname{div} \vec{u}(\vec{r},\, t)$$

8. Plane Monochromatic Waves.

By definition, plane waves have planar wavefronts moving in a specific direction at the wave speed "c", and in particular, monochromatic plane waves are those in which only one angular frequency component "ω" is present.

Let $\hat{\boldsymbol{n}} = n_x \hat{\boldsymbol{\imath}} + n_y \hat{\boldsymbol{\jmath}} + n_z \hat{\boldsymbol{k}}$ be a unit vector in the propagation direction, and $\vec{k} = (\omega/c)\hat{\boldsymbol{n}}$ be the vector wave number, so that $\left\langle \vec{k},\, \vec{r} \right\rangle = \frac{\omega}{c}\left\{x n_x + y n_y + z n_z\right\}$, and then $\left[\langle \vec{k},\, \vec{r} \rangle - \omega t\right]$ is "the phase" of the plane wave at any time "t".

Plane monochromatic elastic waves can be reflected and refracted at the boundary between two different elastic media. If a purely transversal or purely longitudinal wave is incident on such a surface of separation, the result is a mixed wave, containing both transverse and longitudinal parts.

When the direction of propagation is $\hat{\boldsymbol{n}} = \hat{\boldsymbol{\imath}} = (1,\, 0,\, 0)$, then, with $k = \frac{\omega}{c}$, $\left[\left\langle \vec{k},\, \vec{r} \right\rangle - \omega t\right] = [kx - \omega t]$.

9. Surface or Rayleigh Waves.

In this section, the elegant treatment presented in the classical textbook by Landau and Lifshitz, "Theory of Elasticity" second revised and Enlarged Edition, Pergamon Press, 1970, will be followed with some changes in notation and developments in detail [4].

Let the surface of the elastic medium be the Cartesian plane $X0Y$, and suppose the medium fills the half–space $z < 0$.

Let $\vec{u} = u_x\hat{\boldsymbol{\imath}} + u_y\hat{\boldsymbol{\jmath}} + u_z\hat{\boldsymbol{k}}$, **at the free surface** $(z = 0)$, the Boundary Conditions are:

$$\boxed{\mathcal{T}_{zx} = 0, \quad \mathcal{T}_{zy} = 0, \quad \text{and} \quad \mathcal{T}_{zz} = 0, \quad \text{for all time “}t\text{” and all } x \text{ and } y}.$$

Now, from the Helmholtz decomposition $\vec{u} = \vec{u}_L + \vec{u}_T$, we had:

$$\frac{\partial^2 \vec{u}_L}{\partial t^2} - (c_L)^2 \vec{\Delta} \vec{u}_L = 0 \qquad \text{and} \qquad \frac{\partial^2 \vec{u}_T}{\partial t^2} - (c_T)^2 \vec{\Delta} \vec{u}_T = 0,$$

with $\operatorname{curl} \vec{u}_L \equiv 0$, and $\operatorname{div} \vec{u}_T \equiv 0$. **Consider monochromatic waves.** Thus, let

$$u_{0L}(\vec{r}) \equiv e^{i\omega t} \vec{u}_L(\vec{r}, t) \qquad \text{and} \qquad u_{0T}(\vec{r}) \equiv e^{i\omega t} \vec{u}_T(\vec{r}, t),$$

so that:

$$\operatorname{curl} \vec{u}_{0L} \equiv 0, \qquad \text{and} \qquad \operatorname{div} \vec{u}_{0T}(\vec{r}) \equiv 0.$$

Also

$$\vec{u}_{0L}(\vec{r}) + \vec{v}_{0T}(\vec{r}) = e^{i\omega t}\left[\vec{u}_L(\vec{r}, t) + \vec{u}_T(\vec{r}, t)\right] = e^{i\omega t} \vec{u}(\vec{r}, t) \equiv e^{i\omega t} \vec{u}_0(\vec{r}) e^{-i\omega t} = \vec{u}_0(\vec{r}),$$

and **the monochromatic body wave**

$$\vec{u}(\vec{r}, t) = u_0(\vec{r}) e^{-i\omega t} = [\vec{u}_{0L}(\vec{r}) + \vec{u}_{0T}(\vec{r})]\, e^{-i\omega t} = \vec{u}(x, y, z, t), \quad z \leq 0$$

It is required that at the plane boundary surface $(z = 0)$, **a plane monochromatic wave** with a phase $[kx - \omega t]$ be observed, so that

$$\vec{u}(x, y, 0, t) = \vec{u}_0(x, y, 0) e^{-i\omega t} = (\text{constant}) e^{i(kx - \omega t)},$$

and $\vec{u}_0(x, y, 0) = (\text{constant}) e^{ikx}$.

Here, "k", **will be the observed surface wave number**, and $c = \frac{\omega}{k}$, will denote **the observed monochromatic surface wave speed.**

It is also required that $\vec{u}_0(\vec{r}) = \vec{u}_0(x,\, y, z)$ does not depend upon "y", so that

$$\vec{u}_0(\vec{r}) = \vec{u}_0(x,\, z) = \vec{u}_{0L}(x,\, z) + \vec{u}_{0T}(x,\, z), \qquad \vec{u}(\vec{r},\, t) = \left[\vec{u}_{0L}(x,\, z) + \vec{u}_{0T}(x,\, z)\right] e^{-i\omega t}.$$

By definition of monochromatic $\vec{u}_L$ and $\vec{u}_T$ plane waves:

$$\vec{u}_L(\vec{r},\, t) = \vec{u}_{0L}(x,\, z)\, e^{-i\omega t} \qquad \text{and} \qquad \vec{u}_T(\vec{r},\, t) = \vec{u}_{0T}(x,\, z)\, e^{-i\omega t},$$

and from

$$\frac{\partial^2 \vec{u}_L}{\partial t^2} = (c_L)^2 \vec{\nabla}^2 \vec{u}_L, \qquad \frac{\partial^2 \vec{u}_T}{\partial t^2} = (c_T)^2 \vec{\nabla}^2 \vec{u}_T,$$

one gets, using:

$$\frac{\partial^2 \vec{u}_L}{\partial t^2} = -\omega^2 \vec{u}_L, \qquad \frac{\partial^2 \vec{u}_T}{\partial t^2} = -\omega^2 \vec{u}_T,$$

two "Elastic Helmholtz" equations:

$$(c_L)^2 \vec{\nabla}^2 \vec{u}_L + \omega^2 \vec{u}_L = 0 \qquad \text{and} \qquad (c_T)^2 \vec{\nabla}^2 \vec{u}_T + \omega^2 \vec{u}_T = 0$$

or

$$\boxed{\vec{\Delta} \vec{u}_L + \left(\frac{\omega}{c_L}\right)^2 \vec{u}_L = 0} \qquad \text{and} \qquad \boxed{\vec{\Delta} \vec{u}_T + \left(\frac{\omega}{c_T}\right)^2 \vec{u}_T = 0} \tag{14}$$

Guided by the "skin effect" or exponential damping of electromagnetic monochromatic waves under penetration into the interior of a solid conductor filling a half – space $z < 0$, it in natural to seek solutions with exponentially decaying vector factors of the form:

$$\vec{u}_L(x,\, z,\, t) = \vec{f}_L(z)\, e^{i(kx - \omega t)} = u_L x \hat{\boldsymbol{\imath}} + u_L y \hat{\boldsymbol{\jmath}} + u_L z \hat{\boldsymbol{k}},$$

and in Cartesian Coordinates,

$$\vec{\Delta} \vec{u} \equiv \vec{\Delta}(u_x \hat{\boldsymbol{\imath}} + u_y \hat{\boldsymbol{\jmath}} + u_z \hat{\boldsymbol{k}}) = (\Delta u_x)\, \hat{\boldsymbol{\imath}} + (\Delta u_y)\, \hat{\boldsymbol{\jmath}} + (\Delta u_z)\, \hat{\boldsymbol{k}}$$

so that

$$\vec{\Delta}\vec{u}_L = (\Delta u_{Lx})\,\hat{\boldsymbol{\imath}} + \left(\Delta u_{Ly}\right)\hat{\boldsymbol{\jmath}} + (\Delta u_{Lz})\,\hat{\boldsymbol{k}},$$

and

$$\vec{f}_L = (f_{Lx})\,\hat{\boldsymbol{\imath}} + \left(f_{Ly}\right)\hat{\boldsymbol{\jmath}} + (f_{Lz})\,\hat{\boldsymbol{k}}$$

$\Rightarrow$

$$\vec{\Delta}\vec{u}_L = \left\{\left(\Delta f_{Lx}(z)\,e^{ikx}\right)\hat{\boldsymbol{\imath}} + \left(\Delta f_{Ly}(z)\,e^{ikx}\right)\hat{\boldsymbol{\jmath}} + \left(\Delta f_{Lz}(z)\,e^{ikx}\right)\hat{\boldsymbol{k}}\right\} e^{-i\omega t}.$$

But

$$\Delta f_{Lx}(z)\,e^{ikx} = -k^2 f_{Lx}(z)\,e^{ikx} + \frac{d^2}{dz^2} f_{Lx}(z)\,e^{ikx}$$

and (14), implies

$$\left\langle \vec{\Delta}\vec{u}_L,\, \hat{\boldsymbol{\imath}}\right\rangle + \left(\tfrac{\omega}{c_L}\right)^2 \langle \vec{u}_L,\, \hat{\boldsymbol{\imath}}\rangle = 0$$

implies

$$e^{-i\omega t}\left\{-k^2 f_{Lx}(z)\,e^{ikx} + \frac{d^2}{dz^2} f_{Lx}(z)\,e^{ikx}\right\} + \left(\tfrac{\omega}{c_L}\right)^2 f_{Lx}(z)\,e^{i(kx-\omega t)}, \quad \text{etc.}$$

Then get the second order ordinary differential equation for $f_{Lx}(z)$:

$$\frac{d^2}{dz^2} f_{Lx} = \left[k^2 - \left(\tfrac{\omega}{c_L}\right)^2\right] f_{Lx} \tag{15}$$

For waves that are damped inside the body ($z < 0$), it must be $\left[k^2 - \left(\tfrac{\omega}{c_L}\right)^2\right] > 0$, and from the two linearly independent solutions of (15), a solution of the form

$$f_{Lx}(z) = (\text{constant})\,e^{(K_L)z}, \qquad \text{with} \qquad \boxed{K_L = +\sqrt{k^2 - \left(\tfrac{\omega}{c_L}\right)^2}}$$

must be chosen, since $z < 0$ inside the elastic half–space.

Then, the following expression for $u_{Lx}(x,\, z,\, t)$ has been found:

$$u_{Lx}(x,\, z,\, t) = f_{Lx}(z)\,e^{i(kx-\omega t)}, \qquad \Rightarrow \qquad \boxed{u_{Lx}(x,\, z,\, t) = (\text{constant})_x\, e^{(K_L)z}\, e^{i(kx-\omega t)}},$$

and similar expression for $u_{Ly}(x,\, z,\, t)$ and $u_{Lz}(x,\, z,\, t)$.

Writing now

$$\vec{u}_{T}(x,\, z,\, t) = \vec{f}_{T}(z)\, e^{i(kx - \omega t)} = u_{Tx}\hat{\boldsymbol{\imath}} + u_{Ty}\hat{\boldsymbol{\jmath}} + u_{Tz}\hat{\boldsymbol{k}} \qquad \text{and} \qquad \vec{f}_{T} = (f_{Tx})\,\hat{\boldsymbol{\imath}} + \left(f_{Ty}\right)\hat{\boldsymbol{\jmath}} + (f_{Tz})\,\hat{\boldsymbol{k}}$$

a similar analysis leads to:

$$f_{Tx}(z) = (\text{constant})_x\, e^{(K_T)z}, \qquad \text{with} \qquad \boxed{K_T = +\sqrt{k^2 - \left(\frac{\omega}{c_T}\right)^2}},$$

and we must ask $k^2 < \left(\frac{w}{c_T}\right)^2$.

$$u_{Tx}(x,\, z,\, t) = f_{Tx}(z)\, e^{i(kx - \omega t)}, \qquad \Rightarrow \qquad \boxed{u_{Tx}(x,\, z,\, t) = (\text{constant})_x\, e^{(K_T)z}\, e^{i(kx - \omega t)}},$$

and similar expressions for $u_{Ty}(x,\, z,\, t)$ and $u_{Tz}(x,\, z,\, t)$.

It is seen that the parameters K_L and K_T determine the rapidity of the damping of $\vec{u}_L$ and $\vec{u}_T$ towards the interior of the body.

Now it will be shown that $u_y = u_{Ly} + u_{Ty} \equiv 0$, so that

$$\vec{u} = u_x\hat{\boldsymbol{\imath}} + u_2\hat{\boldsymbol{k}} = (u_{Lx} + u_{Tx})\hat{\boldsymbol{\imath}} + (u_{Lz} + u_{Tz})\hat{\boldsymbol{k}},$$

with no y–dependence.

Recall that

$$\varepsilon_{zy} = \frac{1}{2}\left[\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right] = \frac{1}{2}\left(\frac{\partial u_y}{\partial z}\right) = \frac{1}{2\mu}\mathcal{T}_{zy}$$

so that

$$\boxed{\frac{\partial u_y}{\partial z}(x,\, 0,\, t) = \tfrac{1}{\mu}\mathcal{T}_{zy}(x,\, 0,\, t) \equiv 0, \qquad \text{for all } t \text{ and } x}.$$

This Boundary Condition for $\mathcal{T}_{zy}$ will be satisfied if $u_y = (u_{Ly} + u_{Ty})$ satisfies the above identity.

Now

$$u_{Ly}(x,\, z,\, t) = (\text{constant})_y\, e^{(K_L)z}\, e^{i(kx - \omega t)}, \quad \text{and} \quad u_{Ty}(x,\, z,\, t) = (\text{constant})_y\, e^{(K_T)z}\, e^{i(kx - \omega t)} \tag{16}$$

then

$$\frac{\partial u_y}{\partial z} = (\text{constant})_y[K_{_L} + K_{_T}]e^{i(kx-\omega t)} \quad \text{at} \;\; z = 0 \tag{17}$$

and

$$\frac{\partial u_y}{\partial z}(x,\, z,\, t) = \frac{\partial u_{_{Ly}}}{\partial z}(x,\, z,\, t) + \frac{\partial u_{_{Ty}}}{\partial z}(x,\, z,\, t) = 0, \quad \forall t, x \quad \text{at} \;\; z = 0,$$

since $(\text{cosntant})_y = 0$ because $(K_{_L} + K_{_T}) > 0$, and $(\text{constant}\ y)\big[K_{_L} + K_{_T}\big] = 0, \;\Rightarrow\; (\text{constant})_y = 0.$

Next,

$$u_y(x,\, z,\, t) = u_{_{Ly}}(x,\, z,\, t) + u_{_{Ty}}(x,\, z,\, t) = (\text{cosntant})_y e^{i(kx-\omega t)}\left[e^{(K_{_L})z} + e^{(K_{_T})z}\right] \equiv 0,$$

since $(\text{constant})_y = 0$, so that $u_y(x,\, z,\, t) \equiv 0$, and the vector displacement

$$\vec{u}(x,\, z,\, t) = u_x(x,\, z,\, t)\boldsymbol{\hat{\imath}} + u_z(x,\, z,\, t)\boldsymbol{\hat{k}},$$

necessarily must be in the plane $X0Z$ when $\mathcal{T}_{zy} = 0$ at the free surface $z = 0$, and the same is true for $\vec{u}_{_L}(x,\, z,\, t)$ and $\vec{u}_{_T}(x,\, z,\, t)$ due to (16), (17) so that:

$$\vec{u}_{_L}(x,\, z,\, t) = u_{_{Lx}}(x,\, z,\, t)\boldsymbol{\hat{\imath}} + u_{_{Lz}}(x,\, z,\, t)\boldsymbol{\hat{k}}, \qquad \text{and} \qquad \vec{u}_{_T}(x,\, z,\, t) = u_{_{Tx}}(x,\, z,\, t)\boldsymbol{\hat{\imath}} + u_{_{Tz}}(x,\, z,\, t)\boldsymbol{\hat{k}}.$$

The transversal part $\vec{u}_{_T}$ satisfies $\operatorname{div} \vec{u}_{_T} = 0 = \left(\dfrac{\partial u_{_{Tx}}}{\partial x} + \dfrac{\partial u_{_{Tz}}}{\partial z}\right)$, $\forall\, x, z, t.$

Now

$$u_{_{Tz}} = (\text{constant})_z e^{ikx} e^{-i\omega t} e^{(K_{_T})z} \qquad \text{and} \qquad u_{_{Tx}} = (\text{constant})_x e^{ikx} e^{-i\omega t} e^{(K_{_T})z},$$

where $K_{_T} = \sqrt{k^2 - \left(\frac{\omega}{c_T}\right)^2}$.

The divergence–free condition implies:

$$(ik)u_{_{Tx}} + (K_{_T})u_{_{Tz}} = 0, \qquad \text{or} \qquad \frac{u_{_{Tx}}}{u_{_{Tz}}} = -\frac{K_{_T}}{ik}$$

and then,

$$u_{_{Tx}} = (K_{_T})Ae^{ikx - i\omega t + (K_{_T})z}, \qquad u_{_{Tz}} = (-ik)Ae^{ikx - i\omega t + (K_{_T})z},$$

where "A" is some constant.

On the other hand, **the longitudinal part** $\vec{u}_{_L}$, satisfies curl $\vec{u}_{_L} = 0$, or equivalently,

$$\left(\frac{\partial u_{_{Lx}}}{\partial z} - \frac{\partial u_{_{Lz}}}{\partial x}\right) = 0, \quad \forall x,\, z,\, t.$$

But

$$u_{_{Lz}} = (\text{constant})_z^*\, e^{ikx}\, e^{-i\omega t}\, e^{(K_{_L})z} \qquad \text{and} \qquad u_{_{Lx}} = (\text{constant})_x^*\, e^{ikx}\, e^{-i\omega t}\, e^{(K_{_L})z}$$

where $K_{_L} = \sqrt{k^2 - \left(\frac{\omega}{c_L}\right)^2}$.

The curl–free conditions implies:

$$(K_{_L})u_{_{Lx}} - (ik)u_{_{Lz}} = 0, \qquad \text{or} \qquad \frac{u_{_{Lx}}}{u_{_{Lz}}} = \frac{ik}{K_{_L}} = -\frac{k}{iK_{_L}}$$

so that:

$$u_{_{Lx}} = (k)B\,e^{ikx - i\omega t + (K_{_L})z}, \qquad u_{_{Lz}} = (-iK_{_L})B\,e^{ikx - i\omega t + (K_{_L})z}$$

where "B" is some constant.

Now

$$\varepsilon_{zx} = \frac{1}{2\mu}\mathcal{T}_{zx} = 0 = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \text{ at } z = 0, \qquad \text{and} \qquad \nu(\varepsilon_{xx} + \varepsilon_{yy}) + (1-\nu)\varepsilon_{zz} = 0$$

since $\mathcal{T}_{zz} = 0$ at $z = 0$.

From Hooke's law:

$$\begin{aligned}\mathcal{T}_{zz} &= (\lambda + 2\mu)\varepsilon_{zz} + \lambda(\varepsilon_{xx} + \varepsilon_{yy}) \\ &= (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right);\end{aligned}$$

but

$$\frac{\partial u_y}{\partial y} = 0 \qquad \text{and} \qquad \nu = \frac{\lambda}{2(\lambda+\mu)} \quad \Rightarrow \quad (1-\nu) = \frac{\lambda + 2\mu}{2(\lambda+\mu)},$$

then

$$\begin{aligned}\frac{\mathcal{T}_{zz}}{2(\lambda+2\mu)} &= (1-\nu)\frac{\partial u_z}{\partial z} + \frac{\lambda}{2(\lambda+2\mu)}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) \\ &= (1-\nu)\varepsilon_{zz} + \nu(\varepsilon_{xx} + \varepsilon_{yy}) \\ &= (1-\nu)\frac{\partial u_z}{\partial z} + \nu\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right)\end{aligned}$$

$$(c_{_L})^2 - 2\,(c_{_T})^2 = \frac{\lambda + 2\mu}{\rho} - 2\left(\frac{\mu}{\rho}\right) = \frac{\lambda}{\rho}$$

implies

$$\frac{\mathcal{T}_{zz}}{\rho} = \frac{\lambda + 2\mu}{\rho}\left(\frac{\partial u_z}{\partial z}\right) + \frac{\lambda}{\rho}\left(\frac{\partial u_x}{\partial x}\right) = \left[(c_L)^2 - 2(c_{_T})^2\right]\frac{\partial u_x}{\partial x} + (c_{_L})^2\frac{\partial u_z}{\partial z}.$$

Having already applied $\mathcal{T}_{zy} = 0$ on $z = 0$, there remain **two Boundary Conditions**.

The first boundary condition,

$$\mathcal{T}_{zx} = 0 = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), \quad \text{at} \quad z = 0,$$

will be applied to our displacement $\vec{u} = u_x\hat{\boldsymbol{\imath}} + u_z\hat{\boldsymbol{k}} = \vec{u}_{_T} + \vec{u}_{_L}$, with:

$$u_x = (u_{_{Tx}} + u_{_{Lx}}) = e^{ikx - i\omega t}\left[(K_{_T})Ae^{K_{_{Tz}}} + (k)Be^{K_{_{Lz}}}\right]$$

and

$$u_z = (u_{_{Tz}} + u_{_{Lz}}) = e^{ikx - i\omega t}(-i)\left[(k)Ae^{K_{_{Tz}}} + (K_{_L})Be^{K_{_{Lz}}}\right]$$

$$\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = e^{ikx - i\omega t}\left\{\left[(K_{_T})^2Ae^{K_{_{Tz}}} + (kK_{_L})Be^{K_{_{Lz}}}\right] + (-i)(ik)\left[(k)Ae^{K_{_{Tz}}} + (K_{_L})Be^{K_{_{Lz}}}\right]\right\}.$$

But at $z = 0$, $\quad e^{K_{_{Tz}}} = e^{K_{_{Lz}}} = 1$, so that $\mathcal{T}_{zx} = 0 \quad \text{at} \quad z = 0, \;\Rightarrow$

$$\boxed{A\left[(K_{_T})^2 + (k)^2\right] + 2B(kK_{_L}) = 0}. \tag{18}$$

The second boundary condition,

$$\mathcal{T}_{zz} = \rho\left\{\left[(c_{_L})^2 - 2(c_{_T})^2\right]\frac{\partial u_x}{\partial x} + (c_{_L})^2\frac{\partial u_z}{\partial z}\right\} = 0,$$

$\Rightarrow$

$$e^{ikx - i\omega t}\left\{(c_{_L})^2(-i)\left[(k)(K_{_T})Ae^{K_{_{Tz}}} + (K_{_L})^2Be^{K_{_{Lz}}}\right] + \cdots\right.$$
$$\left.\cdots + \left[(c_{_L})^2 - 2(c_{_T})^2\right](ik)\left[(K_{_T})Ae^{K_{_{Tz}}} + (k)Be^{K_{_{Lz}}}\right]\right\} = 0, \quad \text{at} \quad z = 0$$

$\Rightarrow$

$$2A(c_{_T})^2kK_{_T}+B\Big\{(c_{_L})^2\left[(K_{_L})^2-k^2\right]+2(c_{_T})^2k^2\Big\}=0.$$

This second linear equation for A and B, can be simplified, **firstly**, by dividing it by $(c_{_T})^2$:

$$2AkK_{_T}+B\left\{\left(\tfrac{c_L}{c_T}\right)^2\left[(K_{_L})^2-k^2\right]+2k^2\right\}=0$$

Secondly, transforming the expression $\left[(K_{_L})^2-k^2\right]$.

By definition of $K_{_L}$,

$$\left[(K_{_L})^2-k^2\right]=k^2-\frac{\omega^2}{(c_{_L})^2}-k^2=-\frac{\omega^2}{(c_{_L})^2}.$$

Also, by definition of $K_{_T}$,

$$(K_{_T})^2=k^2-\frac{\omega^2}{(c_{_T})^2},\quad\Rightarrow\quad(c_{_T})^2(K_{_T})^2=k^2(c_{_T})^2-\omega^2\quad\Rightarrow\quad\left(\tfrac{c_T}{c_L}\right)^2(K_{_T})^2=k^2\left(\tfrac{c_T}{c_L}\right)^2-\frac{\omega^2}{(c_{_L})^2}$$

$\Rightarrow$

$$\frac{-\omega^2}{(c_{_L})^2}=\left(\tfrac{c_T}{c_L}\right)^2(K_{_T})^2-k^2\left(\tfrac{c_T}{c_L}\right)^2=\left(\tfrac{c_T}{c_L}\right)^2\left[(K_{_T})^2-k^2\right]=-\left(\tfrac{c_T}{c_L}\right)^2\left[k^2-(K_{_T})^2\right]$$

then

$$\left[(K_{_L})^2-k^2\right]=-\left(\tfrac{c_T}{c_L}\right)^2\left[k^2-(K_{_T})^2\right].$$

Substituting into the homogeneous equation:

$$2AkK_{_T}+B\left\{-\left[k^2-(K_{_T})^2\right]+2k^2\right\}=2AkK_{_T}+B\left[k^2+(K_{_T})^2\right]=0\tag{19}$$

The homogeneous algebraic system (18) and (19) for the unknown constants A and B is compatible under the zero-determinant condition:

$$\left[k^2+(K_{_T})^2\right]^2-4k^2K_{_T}K_{_L}=0$$

or, squaring and substituting the values of $(K_{_T})^2$ and $(K_{_L})^2$:

$$\left[2k^2-\left(\tfrac{\omega}{c_T}\right)^2\right]^4=16k^4\left[k^2-\left(\tfrac{\omega}{c_T})\right)^2\right]\left[k^2-\left(\tfrac{\omega}{c_L}\right)^2\right].\tag{20}$$

From this polynomial equation **we obtain the relation between ω and k** so that at $z = 0$, we get **a surface displacement wave** $\vec{u}(x,\, 0,\, t)$, being the limiting value of $\vec{u}(x,\, z,\, t) = u_x(x,\, z,\, t)\hat{\boldsymbol{\imath}} + u_z(x,\, z,\, t)\hat{\boldsymbol{k}}$ satisfying the free stress Boundary Condition at $z = 0$, and having the form:

$$\vec{u}(x,\, 0,\, t) = e^{i(kx - \omega t)} \Big\{ [(K_{_T})A + kB]\hat{\boldsymbol{\imath}} - (i)[(k)A + (K_{_L})B]\hat{\boldsymbol{k}} \Big\}$$

Thinking of "k" as an unknown for an equation of the eight degree, it is convenient to change from (ω/k) to an auxiliary unknown ξ defined by:

$$\boxed{\left(\frac{\omega}{k}\right) = (c_{_T})\xi} \qquad \text{or} \qquad \left(\frac{\omega}{c_{_T}}\right) = k\xi$$

It is easy to see that k^8 cancels from equation (20). Expanding it:

$$16k^8 - 32k^6\left(\tfrac{\omega}{c_T}\right)^2 + 24k^4\left(\tfrac{\omega}{c_T}\right)^4 - 16k^2\left(\tfrac{\omega}{c_T}\right)^6 + (\omega c_{_T})^8 = \cdots$$

$$\cdots \;=\; 16k^4\left\{k^4 - k^2\left[\left(\tfrac{\omega}{c_T}\right)^2 + \left(\tfrac{\omega}{c_L}\right)^2\right] + \left(\tfrac{\omega}{c_T}\right)^2\left(\tfrac{\omega}{c_L}\right)^2\right\}$$

$$= \; 16k^8 - 16k^6\left[\left(\tfrac{\omega}{c_T}\right)^2 + \left(\tfrac{\omega}{c_L}\right)^2\right] + 16k^4\left(\tfrac{\omega}{c_T}\right)^2\left(\tfrac{\omega}{c_L}\right)^2$$

with the above substitution, we get:

$$\xi^6 - 8\xi^4 + 8\xi^2\left[3 - 2\left(\tfrac{c_T}{c_L}\right)^2\right] - 16\left[1 - \left(\tfrac{c_T}{c_L}\right)^2\right] = 0$$

(see page 32).

We see that ξ dependes only on the ratio $\left(\frac{c_T}{c_L}\right)$, which in turn depends only on Poisson's ratio ν:

$$\left(\frac{c_{_T}}{c_{_L}}\right) = \sqrt{\frac{1 - 2\nu}{2(1 - \nu)}},$$

(see page 21, equations (11) and (12)), ξ must be real and positive, and also $\xi < 1$ (so that $K_{_T}$ and $K_{_L}$ are real).

Since our algebraic equation contains only even powers of ξ, the substitution $\xi^2 = \eta$ **leads to a cubic equation in** η, which has only one root satisfying $\xi > 0$ and $\xi < 1$, and so, a single value of ξ is obtained for any given value of $\left(\frac{c_T}{c_L}\right)$.

Therefore, for both surface and volumen waves, the frequency ω is proportional to the wave number k, and the coefficient of proportionality is the velocity U of propagation of the wave, $U = c_{_T}\xi$ (and since $\xi c_{_T} = (\omega/k) \equiv c, \quad U = c$).

This gives the velocity of propagation of surface waves in terms of the velocities c_T and c_L of the transverse and longitudinal volume waves. The ratio of the amplitudes of the transverse and longitudinal parts of the wave is given in terms of ξ by the formula:

$$\frac{A}{B} = -\frac{2-\xi^2}{2\sqrt{1-\xi^2}}, \qquad \text{(see page 34, equation (21)).}$$

The ratio $\left(\frac{c_T}{c_L}\right)$ actually varies from $1/\sqrt{2}$ to 0 when ν varies from 0 to (1/2), and then ξ varies from 0.874 to 0.955, so that c $= \xi\, c_T < c_T$.

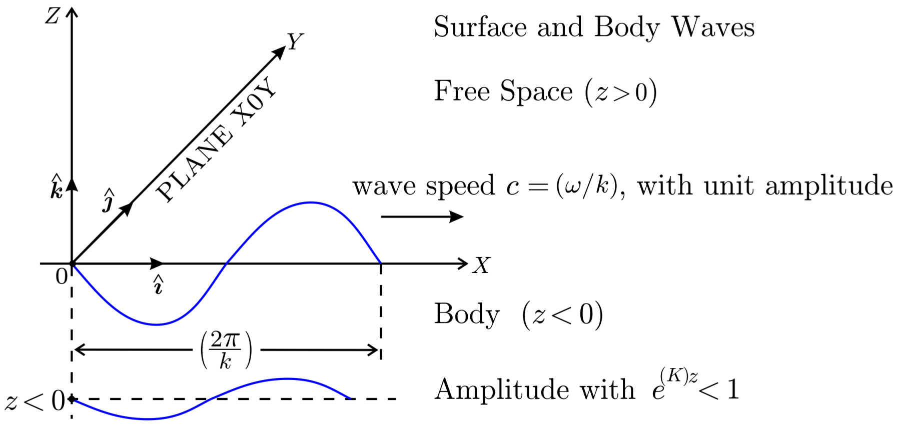


Figure 10. Rayleigh and body waves.

10. Deduction of the Sixth–Degree Equation for ξ.

The fully developed algebraic equation is, after canceling $16k^8$:

$$k^6\left\{-16\left(\frac{\omega}{c_T}\right)^2+16\left(\frac{\omega}{c_L}\right)^2\right\}+k^4\left\{24\left(\frac{\omega}{c_T}\right)^4-16\left(\frac{\omega}{c_T}\right)^2\left(\frac{\omega}{c_L}\right)^2\right\}-8k^2\left(\frac{\omega}{c_T}\right)^6+\left(\frac{\omega}{c_T}\right)^8=0$$

Now use $k=\frac{1}{\xi}\left(\frac{\omega}{c_T}\right), \quad k^2=\left(\frac{1}{\xi}\right)^2\left(\frac{\omega}{c_T}\right)^2, \quad$ etc.

$$\left(\frac{1}{\xi}\right)^6\left(\frac{\omega}{c_T}\right)^6\left\{-16\left(\frac{\omega}{c_T}\right)^2+16\left(\frac{\omega}{c_L}\right)^2\right\}+\left(\frac{1}{\xi}\right)^4\left(\frac{\omega}{c_T}\right)^4\left\{24\left(\frac{\omega}{c_T}\right)^4-16\left(\frac{\omega}{c_T}\right)^2\left(\frac{\omega}{c_L}\right)^2\right\}-\cdots$$

$$\cdots-8\left(\frac{1}{\xi}\right)^2\left(\frac{\omega}{c_T}\right)^2\left(\frac{\omega}{c_T}\right)^6+\left(\frac{\omega}{c_T}\right)^8=0$$

then,

$$-16\left(\frac{1}{\xi}\right)^6\left(\frac{\omega}{c_T}\right)^8+16\left(\frac{1}{\xi}\right)^6\left(\frac{\omega}{c_T}\right)^6\left(\frac{\omega}{c_L}\right)^2+24\left(\frac{1}{\xi}\right)^4\left(\frac{\omega}{c_T}\right)^8-16\left(\frac{1}{\xi}\right)^4\left(\frac{\omega}{c_T}\right)^6\left(\frac{\omega}{c_L}\right)^2-8\left(\frac{1}{\xi}\right)^2\left(\frac{\omega}{c_T}\right)^8+\left(\frac{\omega}{c_T}\right)^8=0$$

Next, divide by $\left(\frac{\omega}{c_T}\right)^8$:

$$-16\left(\tfrac{1}{\xi}\right)^6+16\left(\tfrac{1}{\xi}\right)^6\frac{\left(\frac{\omega}{c_L}\right)^2}{\left(\frac{\omega}{c_T}\right)^2}+24\left(\tfrac{1}{\xi}\right)^4-16\left(\tfrac{1}{\xi}\right)^4\frac{\left(\frac{\omega}{c_L}\right)^2}{\left(\frac{\omega}{c_T}\right)^2}-8\left(\tfrac{1}{\xi}\right)^2+1=0$$

Now, multiply by ξ^6:

$$-16+16\left(\frac{c_T}{c_L}\right)^2+24\xi^2-16\xi^2\left(\frac{c_T}{c_L}\right)^2-8\xi^4+\xi^6=0$$

and finally,

$$-16\left[1-\left(\frac{c_T}{c_L}\right)^2\right]+8\xi^2\left[3-2\left(\frac{c_T}{c_L}\right)^2\right]-8\xi^4+\xi^6=0,$$

and the third degree equation for $\eta=\xi^2$ is:

$$\eta^3-8\eta^2+8\eta\left[3-2\left(\frac{c_T}{c_L}\right)^2\right]-16\left[1-\left(\frac{c_T}{c_L}\right)^2\right]=0.$$

This cubic equation always has one read root, and for this equation, that root must be positive, and also $\eta<1$. For $k>0$ and $\omega>0$, ξ must be positive, so that $0<\xi<1$.

It is seen that the Boundary Conditions $\mathcal{T}_{zx}=0$ and $\mathcal{T}_{zz}=0$, allow to find **two ratios**: (c_T/c_L) (or (K_L/K_T), **and the ratio of amplitudes** (A/B), since there is a one–to–one correspondence between ξ and Poisson's ratio ν, due to the one–to–one between (c_T/c_L) and ξ through the sixth degree polynomial equation for ξ, or the equivalent third degree polynomial equation for $\eta=\xi^2$, and $\xi c_T=(\omega/k)$.

Also, remember that by definition, K_L and K_T are both real constants only when $k^2>\left(\frac{\omega}{c_L}\right)^2$ and $k^2>\left(\frac{\omega}{c_T}\right)^2$, or equivalently, when $\left(\frac{\omega}{k}\right)^2<(c_L)^2$ and $\left(\frac{\omega}{k}\right)^2<(c_T)^2$, and from the second inequality,

$$\left(\frac{\frac{\omega}{k}}{c_T}\right)^2<1 \qquad \text{or} \quad 0<\xi<1,$$

and $c=\frac{\omega}{k}$ implies $c<c_T<c_L$.

The abscense of the imaginary part in a root for k **from the sixth degree equation**, indicates a weak decay of the surface wave caused just by an ordinary spatial wave decay underneath, and that is the reason why the Rayleigh wave can propagate over a large distance along the solid surface; the penetration underneath the surface of body is small. According to Nazarchuk, Z. et al [5], at the wave–length $\lambda=2\pi/k$, the intensity is about 5 % of the intensity on the surface of the body. Also, an approximate value for evaluating the speed c of the Rayleigh surface wave has been given by Yermolov, I. N., et al [6], who gives $c\approx 0.93c_T$. Viktorov, I. A. [7], has presented a short version of the linear combination of longitudinal and transversal harmonic wave vibrations propagation along an unloaded or free plane surface.

Observing that the pair of homogeneous linear equations (18) and (19) for A and B imply

$$\frac{A}{B} = \frac{-2kK_L}{(K_T)^2 + k^2} = -\frac{k^2 + (K_T)^2}{2kK_T},$$

it follows that: $\vec{u} = u_x \hat{\boldsymbol{\imath}} + u_z \hat{\boldsymbol{k}}$, with

$$u_x = e^{ikx - i\omega t} B \left[(K_T)\tfrac{A}{B} e^{K_{Tz}} + k e^{K_{Lz}} \right] \quad \text{and} \quad u_z = e^{ikx - i\omega t} (-i) B \left[k\tfrac{A}{B} e^{K_{Tz}} + (K_L) e^{K_{Lz}} \right]$$

Therefore, the vector elastic displacement observed at the surface $z = 0$ is:

$$\vec{u} = B \left\{ \left[(K_T)\tfrac{A}{B} + k \right] \hat{\boldsymbol{\imath}} + (-i) \left[k\tfrac{A}{B} + K_L \right] \hat{\boldsymbol{k}} \right\} e^{ikx - i\omega t}$$

and the constant "B" depends on the intensity of the plane–wave on the free surface, since (A/B) depends only on "k" (or ξ) and the elastic constants and density of the elastic solid half–space.

Also, once ξ in determined in terms of (c_T/c_L), described with ξ^*, and since $(c_T)\xi^* = \frac{\omega}{k}$, then, for each pair of values of c_T and c_L, the relation between ω and k becomes known.

Besides, $c = \frac{\omega}{k}$, so that $\xi = \frac{c}{c_T}$, and the velocity of propagation of the surface wave $c = U = (c_T)\xi$ is also determined once c_T and c_L are known for the elastic half–plane.

Now

$$\zeta^2 = \frac{\left(\frac{\omega}{k}\right)^2}{(c_T)^2} \qquad \Rightarrow \qquad \left(1 - \zeta^2\right) = 1 - \frac{\left(\frac{\omega}{k}\right)^2}{(c_T)^2}, \qquad \Rightarrow \qquad \left(2 - \zeta^2\right) = 2 - \frac{\left(\frac{\omega}{k}\right)^2}{(c_T)^2}$$

But

$$(K_T)^2 = k^2 - \left(\tfrac{\omega}{c_T}\right)^2 \qquad \Rightarrow \qquad (K_T)^2 + k^2 = 2k^2 - \left(\tfrac{\omega}{c_T}\right)^2 = k^2 \left[2 - \zeta^2\right]$$

and

$$kK_T = k\sqrt{k^2 - \left(\tfrac{\omega}{c_T}\right)^2} = k^2\sqrt{1 - \zeta^2},$$

so that

$$\frac{A}{B} = -\frac{\left[k^2 + (K_T)^2\right]}{2kK_T} = -\frac{2 - \zeta^2}{2\sqrt{1 - \zeta^2}} \tag{21}$$

as was said before.

With this values for (A/B), the surface elastic displacement $\vec{u} = u_x\hat{\boldsymbol{\imath}} + u_z\hat{\boldsymbol{k}}$ becomes:

$$u_x = B\,e^{ikx - i\omega t}\left[(K_T)\tfrac{A}{B} + k\right] \qquad \text{and} \quad u_z = (-iB)\,e^{ikx - i\omega t}\left[k\tfrac{A}{B} + (K_L)\right]$$

Since

$$\operatorname{Re}\left\{e^{ikx - i\omega t}\right\} = \cos(kx - \omega t) \qquad \text{and} \qquad \operatorname{Re}\left\{(-i)\,e^{ikx - i\omega t}\right\} = -\sin(kx - \omega t),$$

it follows that:

$$u_x\hat{\boldsymbol{\imath}} + u_z\hat{\boldsymbol{k}} = B\left[(K_T)\tfrac{A}{B} + k\right]\cos(kx - \omega t)\hat{\boldsymbol{\imath}} - B\left[k\tfrac{A}{B} + (K_L)\right]\sin(kx - \omega t)\hat{\boldsymbol{k}}$$

which represents a rotating displacement vector in plane $X0Z$ (which is orthogonal to the free Boundary surface plane $X0Y$), and for each fixed "x", describes an ellipse in the planes that are parallel to plane $X0Z$.

11. Appendix for Momentum Equation in Linear Elasticity.

- $\rho\vec{a} = \nabla \cdot \mathcal{T}, \quad \vec{a} = \ddot{\vec{d}}, \quad \vec{d} = u\hat{\boldsymbol{\imath}} + v\hat{\boldsymbol{\jmath}} + w\hat{\boldsymbol{k}}$, is the tensor general form when $\vec{b} = 0$.

- $\langle \rho\vec{a}, \hat{\boldsymbol{\imath}}\rangle = \rho\ddot{u} = \dfrac{\partial \mathcal{T}_{xx}}{\partial x} + \dfrac{\partial \mathcal{T}_{xy}}{\partial y} + \dfrac{\partial \mathcal{T}_{xz}}{\partial z}; \quad \mathcal{T}_{xy} = 2\mu\varepsilon_{xy}, \quad \mathcal{T}_{xz} = 2\mu\varepsilon_{xz}.$

- $\langle \rho\vec{a}, \hat{\boldsymbol{\jmath}}\rangle = \rho\ddot{v} = \dfrac{\partial \mathcal{T}_{yx}}{\partial x} + \dfrac{\partial \mathcal{T}_{yy}}{\partial y} + \dfrac{\partial \mathcal{T}_{yz}}{\partial z}; \quad \mathcal{T}_{yx} = 2\mu\varepsilon_{yx}, \quad \mathcal{T}_{yz} = 2\mu\varepsilon_{yz}.$

- $\langle \rho\vec{a}, \hat{\boldsymbol{k}}\rangle = \rho\ddot{w} = \dfrac{\partial \mathcal{T}_{zx}}{\partial x} + \dfrac{\partial \mathcal{T}_{zy}}{\partial y} + \dfrac{\partial \mathcal{T}_{zz}}{\partial z}; \quad \mathcal{T}_{zx} = 2\mu\varepsilon_{zx}, \quad \mathcal{T}_{zy} = 2\mu\varepsilon_{zy}.$

- $2\varepsilon_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}, \qquad 2\varepsilon_{xz} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}, \qquad 2\varepsilon_{yz} = \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}.$

- Hooke $\begin{cases} \mathcal{T}_{xx} = \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\varepsilon_{xx}, & \varepsilon_{xx} = \dfrac{\partial u}{\partial x} \\ \mathcal{T}_{yy} = \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\varepsilon_{yy}, & \varepsilon_{yy} = \dfrac{\partial v}{\partial y} \\ \mathcal{T}_{zz} = \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\varepsilon_{zz}, & \varepsilon_{zz} = \dfrac{\partial w}{\partial z} \end{cases}$

- $$\left.\begin{array}{l} \tau_{xx} = \lambda\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}\right)+2\mu\dfrac{\partial u}{\partial x}, \quad \Rightarrow \quad \dfrac{\partial \tau_{xx}}{\partial x} = \lambda\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 v}{\partial y\partial x}+\dfrac{\partial^2 w}{\partial z\partial x}\right)+2\mu\dfrac{\partial^2 u}{\partial x^2} \\ \text{Hooke}\begin{cases} \tau_{xy} \;=\; \mu\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right) \;\Rightarrow\; \dfrac{\partial \tau_{xy}}{\partial y} = \mu\left(\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 v}{\partial x\partial y}\right) \\ \tau_{xz} \;=\; \mu\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\right) \;\Rightarrow\; \dfrac{\partial \tau_{xz}}{\partial z} = \mu\left(\dfrac{\partial^2 u}{\partial z^2}+\dfrac{\partial^2 w}{\partial x\partial z}\right) \end{cases} \end{array}\right\} \Rightarrow$$

- $$\rho\ddot{u} = \left[\lambda\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial x}+\frac{\partial^2 w}{\partial z\partial x}\right)+2\mu\frac{\partial^2 u}{\partial x^2}\right]+\left[\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)\right]+\left[\mu\left(\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial x\partial z}\right)\right].$$

- $$\rho\ddot{u} = (\lambda+\mu)\left(\frac{\partial^2 u}{\partial x^2}\right)+\lambda\left(\frac{\partial^2 v}{\partial y\partial x}+\frac{\partial^2 w}{\partial z\partial x}\right)+\mu\Delta u+\mu\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial z}\right), \quad \text{or}$$

- $$\rho\ddot{u} = (\lambda+\mu)\left[\left(\frac{\partial^2 u}{\partial x^2}\right)+\left(\frac{\partial^2 v}{\partial y\partial x}+\frac{\partial^2 w}{\partial z\partial x}\right)\right]+\mu\Delta u \quad \Rightarrow$$

- $$\rho\left\langle \ddot{\vec{d}},\, \hat{\boldsymbol{\imath}}\right\rangle = \mu\langle\vec{\Delta}\vec{d},\, \hat{\boldsymbol{\imath}}\rangle + (\lambda+\mu)\langle \operatorname{grad}\operatorname{div}\vec{d},\, \hat{\boldsymbol{\imath}}\rangle, \quad \text{because}$$

- $$\operatorname{grad}\operatorname{div}\vec{d} = \operatorname{grad}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) = \hat{\boldsymbol{\imath}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial x}+\frac{\partial^2 w}{\partial z\partial x}\right)+\hat{\boldsymbol{\jmath}}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial z\partial y}\right)+\cdots$$
$$\cdots+\hat{\boldsymbol{k}}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial z^2}\right)$$

- $\left\langle \operatorname{grad}\operatorname{div}\vec{d},\, \hat{\boldsymbol{\imath}}\right\rangle = \left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 v}{\partial y\partial x}+\dfrac{\partial^2 w}{\partial z\partial z}\right)$, and analogous components along $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$

- Also, in Cartesian Coordinates,

$$\vec{\Delta}\vec{d} = (\Delta u)\hat{\boldsymbol{\imath}} + (\Delta v)\hat{\boldsymbol{\jmath}} + (\Delta w)\hat{\boldsymbol{k}}.$$

- Therefore,

$$\boxed{\rho\ddot{\vec{d}} = \mu\vec{\Delta}\vec{d} + (\lambda+\mu)\operatorname{grad}\operatorname{div}\vec{d},}$$

that is **Newton's momentum equation for elastic solids satisfying Hooke's Law.**

12. Appendix for Real Physical Elastic Waves Versus Ideal Mathematical Models.

In the real physical world there is nothing like some elastic solid filling the geometrical domain defined by a Half–Space, as in Landau and Lifshitz [4], elegant treatment of Rayleigh Surface Elastic Waves.

Seismic waves propagate through the crust of the Earth, which is NOT homogeneous, being composed of several layers differing in their elastic constants (see reference [8]), and only approximately spherical, and even if the wave fronts were spherical, they could not be considered as Plane Waves at all.And this, without considering some sharp discontinuities in the geological substratum, present in our planet.

However, depending upon the distance from the source or sources originating initially the harmonic vibrations, exhibited by these real waves, it is sometimes reasonable to approximate the wave fronts by planes orthogonal to a main direction of wave propagation. Besides, the real, physical source, forcing the crust elastic vibrations, does not act throughout all time, as happens in the explosions artificially caused on the Earth's surface in order to explore underground oil reservoirs.

Despite these limitations, some simpler mathematical models help to start a qualitative understanding of some basic features also found in the observed real, physical waves, and the same applies to the use of concentrated forces represented by means of the Dirac delta distribution, even though no such "force concentrated in a point and with infinite tension value" exists in nature, but it has turned into a very fruitful modelling in many areas of mathematical physics.

There is an enormous technical literature on seismic waves, and that it is completely outside of the scope of the present introductory report, having only the purpose of providing a relatively quick overview for the relevance to elastic waves , of the high-order mimetic numerical methods, both in space and time, improving faithfulness to the physics of the real problems under modelling.

13. Appendix for the Linear Poroelastic Models of pore Fluid Pressure Interacting with Mechanical Elastic Deformation of Porous Rocks.

For the history and diverse related problems of interest, details can be found in the article by Detourmay, E. and Cheng, A.H.-D, "Fundamentals of poroelasticity", chapter 5 in Comprehensive Rock Engineering: Principles, Practice and Projects, Vol. II, Analysis and Design Method, ed. C. Fairhurst, Pergamon Press, pp. 113 - 171, 1993, [9].

In the "Crash Course in Linear Elasticity" this Appendix **will be limited to see** how the already derived classical elastic relations can be extended to cover pore pressure, **and then show how Mimetic Difference Methods can be used to model hydraulic fractures.**

Processes in continuous media usually start with momentum balance or Newton's Law, coupled with some constitutive equations, which in the linearly elastic case, were Hooke's strain–stress relations.

The microstructure of porous rocks has a length scale about 1/100 th of the length scale that underpins the continuum model, so that a system of Partial Differential Equations is still a reasonable model for this particular type of solids, and yet, is small enough to allow the introduction of genuine macroscopic scale material heterogeneity.

The conceptual Biot Model chooses as kinematic quantities the solid displacement vector u_i which tracks the movement of the porous solid with respect to a reference configuration, and a specific discharge vector q_i which describes the motion of the fluid relative to the solid. The specific discharge q_i is formally defined as the rate of fluid volume crossing a unit area of porous solid whose normal is in the x_i direction $i = 1,\, 2,\, 3$. Two "strain" quantities are introduced. One is the usual strain tensor ε_{ij}, and the other is ζ, the variation of fluid content, that is, the variation of fluid volume per unit volume of porous material, and the fluid mass balance is described by

$$\frac{\partial \zeta}{\partial t} = -\operatorname{div} q, \quad \text{where} \quad q = (q_1,\, q_2,\, q_3)$$

The basic dynamic variables are the scalar pressure p and the stress tensor $\mathcal{T}_{ij}$. The work increment associated with the strain increment $d\varepsilon_{ij}$ and $d\zeta$, in the presence of the stress $\mathcal{T}_{ij}$ and pore pressure p is:

$$dW = \mathcal{T}_{ij} d\varepsilon_{ij} + p d\zeta \quad \text{(with Einstein's summation convention)}$$

In the Biot model, the shear stress at the contact between fluid and solid, associated with a local velocity gradient in the fluid, is not considered. The pore pressure must be locally equilibrated between neighboring pores over the length scale, and since the time scale and the length scale are linked in a way depending on the viscosity of the interstitial fluid, this requires the consideration of quasi-static processes.

13.1. Poroelastic Constitutive Equations.

Recalling that the classical elastic relations are

$$\mathcal{T}_{ij} = (2\mu)\varepsilon_{ij} = (2G)\varepsilon_{ij} \quad \text{if } i \neq j, \qquad \text{and} \qquad \varepsilon_{xx} = \tfrac{1}{E}\big[\mathcal{T}_{xx} - \nu(\mathcal{T}_{yy} + \mathcal{T}_{zz})\big], \quad \text{etc.}$$

and that $K = E/[3(1-2\nu)]$, so that all cases are covered by the linear relation

$$\varepsilon_{ij} = \frac{\mathcal{T}_{ij}}{26} - \left(\frac{1}{6G} - \frac{1}{9K}\right)\delta_{ij}(\mathcal{T}_{11} + \mathcal{T}_{22} + \mathcal{T}_{33}),$$

its extension to poroelasticity in the case of isotropic materials is:

$$\varepsilon_{ij} = \frac{\mathcal{T}_{ij}}{2G} - \left(\frac{1}{6G} - \frac{1}{9K}\right)\delta_{ij}(\mathcal{T}_{11} + \mathcal{T}_{2} + \mathcal{T}_{33}) + \frac{1}{3H'}\delta_{ij}p, \tag{22}$$

and the combined effect of pore pressure p and the elastic solid stress is:

$$\zeta = \frac{(\mathcal{T}_{11} + \mathcal{T}_{22} + \mathcal{T}_{33})}{3H''} + \frac{p}{R'}. \tag{23}$$

Here, K is the drained bulk modulus. While many material constants are initially introduced, only three are actually independent K, K_u the undrained bulk modulus, and **the Biot coefficient $\boldsymbol{\alpha}$**.

The additional constitutive constants H', H'' and R' characterize the coupling between the solid and fluid stress and strain under the assumption of thermodynamical reversibility, the work increment dW must be an exact differential.

Since

$$dW = \mathcal{T}_{ij} d\varepsilon_{ij} + p d\zeta = \varepsilon_{ij} d\mathcal{T}_{ij} + \zeta dp,$$

then the Euler conditions must hold:

$$\frac{\partial \varepsilon_{ij}}{\partial p} = \frac{\partial \zeta}{\partial \mathcal{T}_{ij}} \tag{24}$$

Combining (24) with (22) and (23), leads to $H'' = H'$. The isotropic constitutive law there fore involves only four constants G, K, H' and R'. H' and R' were originally denoted as H and R in Biot's 1941 paper [10]. Since the same symbols were later redefined [11], the prime superscripts have been added to avoid any confusion.

It is convenient to separate the constitutive equations of an isotropic poroelastic material into.

I) A deviatoric response

$$e_{ij} = \frac{1}{2G} s_{ij}, \quad \text{and}$$

II) A volumetric response

$$\varepsilon = -\left(\frac{P}{K} - \frac{p}{H'}\right), \qquad \zeta = -\left(\frac{P}{H'} - \frac{p}{R'}\right)$$

where s_{ij} and e_{ij} denote the deviatoric stress and strain, P the mean or total pressure (isotropic compressive stress), and ε the volumetric strain.

$$s_{ij} = \mathcal{T}_{ij} + P\delta_{ij}$$

$$e_{ij} = \varepsilon_{ij} - \tfrac{\varepsilon}{3}\delta_{ij}$$

$$P = -\tfrac{1}{3}\big(\mathcal{T}_{11} + \mathcal{T}_{22} + \mathcal{T}_{33}\big)$$

$$\varepsilon = \big(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\big)$$

It is seen that the deviatoric response is purely elastic, while the coupled effects, which involve constants H' and R', appear only in the volumetric stress–strain relation, but this is not true for anisotropic materials.

13.2. Volumetric Response of a Linear Isotropic Poroelastic Material.

There are two modes of response, representing limiting behaviors of the material:

A) The undrained response characterizes the condition where the fluid is trapped in the porous solid such that $\zeta = 0$.

B) The drained response corresponds to zero pore pressure, $p = 0$. This condition can however be relaxed to include any initial pore pressure field that is in equilibrium.

Obviously under the undrained $\zeta = 0$, the pore pressure p will be proportional to the total pressure P:

$$p = BP$$

where **the coefficient** $B = R'/H'$ is known as **the Skempton pore pressure coefficient.** Also, in this case

$$\varepsilon = -\frac{P}{K_u}, \qquad \text{where} \qquad K_u = K\left(1 + \frac{KR'}{(H')^2 - KR'}\right)$$

is the undrained bulk modulus of the material.

Under the drained condition $p = 0$ one gets:

$$\varepsilon = -\frac{P}{K}$$

Thus, under both the drained and undrained conditions, the poroelastic material behaves as an elastic one, and since $K_u > K$ one can say that the undrained material is stiffer than the drained one.

Substituting $P = -K\varepsilon$, one gets under the drained condition:

$$\zeta = \alpha\varepsilon$$

where $\alpha = \dfrac{K}{H'} =$ **Biot Coefficient**.

This equation gives a meaning to the constant α as the ratio of the fluid volume gained (or lost) in a material element to the volume change of that element, when the pore pressure is allowed to return to its initial state. Also it is seen that $\alpha \leq 1$, since the volume of fluid gained (or lost) by an element cannot be grater than the total volume change of that element (under the linearized approximation).

The three volumetric constitutive constants K, K_u and α, which will be chosen in place of K, H' and R' as the basic set, have thus physical meanings that are associated with the drained and undrained responses of the material, and the range of variations are: [0, 1] for α and $[K, \infty]$ for K_u.

Having thus chosen our set of three basic constants α, K and K_u, the volumetric relations can be rewritten as:

$$\begin{aligned}\varepsilon &= -\frac{1}{K}(P - \alpha p)\\ \zeta &= -\frac{\alpha}{K}\left(P - \frac{p}{B}\right), \qquad \text{where} \quad B = \frac{K_u - K}{\alpha K_u}\end{aligned}$$

Inverting these relations, get:

$$P = \alpha M \zeta - K_u \varepsilon$$

$$p = M(\zeta - \alpha\varepsilon), \qquad \text{where} \quad M = \frac{K_u - K}{\alpha^2} = \frac{(H')^2 R'}{(H')^2 - K R'}$$

The constant M is sometimes called **the Biot modulus**, and it is the inverse of a "storage coefficient", that is the increase of the amount of fluid (per unit volume of rock) as a result of a unit increase of pore pressure, under constant volumetric strain, and since $P = P(p,\, \varepsilon)$, so that $\zeta = \zeta(P, p) = \zeta(P(p, \varepsilon), p)$, one can write:

$$\frac{1}{M} = \left.\frac{\partial \zeta}{\partial p}\right|_{\varepsilon}$$

13.3. Micromechanical Approach.

While the previous constitutive model so far discussed, describes the response of a porous material as a whole, without explicitly taking into account the individual contribution of its solid and fluid constituents, a more refined approach can be carried out at the cost of more measurements, and thus gaining additional insight to the interaction among the constituents. The "loading" of the material is now considered as a combination $\{P,\, p\}$ of independent P and p.

Following Terzaghi, an alternative to the loading decomposition $\{P,\, p\}$ will also be considered, namely $\{P',\, p'\}$, where:

$$P' \;=\; P - p \quad \text{(a Terzaghi “effective pressure”)}$$

$p' = p$, i.e., a confining pressure and a pore pressure of the same magnitude p. The loading $\{P',\, p'\}$ is denoted as "Π – loading".

13.4. Volumetric Response of Fluid – Infiltrated Porous Solids.

Consider a sample of porous material of volume V, containing an **interconnected** pore space of volume V_p (this looks "labyrinthic" under microscope). The combined volume of the solid phase and **isolated pores** is denoted by V_s, with $V = V_p + V_s$. Assuming full saturation, the volume of fluid which can **freely** circulate in the sample is $V_f = V_p$.

The porosity ϕ is defined as the ratio V_p/V. Under a loading $\{P,\, p\}$, there will be volumetric increments ΔV and ΔV_p.

Invoking linearity between stress and strain, one gets the relations

$$\frac{\Delta V}{V} = -\frac{1}{K}(P - \alpha p) \qquad \frac{\Delta V_p}{V_p} = -\frac{1}{K_p}(P - \beta p),$$

where K_p is the bulk modulus for the pore volumetric strain and β is a dimensionless effective stress coefficient. This approach will not be pursued here.

13.5. Transport Law.

The fluid transport in the intersticial space is describe by Darcy's law:

$$q_i = -k\left(\frac{\partial p}{\partial x_i} - f_i\right), \qquad \text{with } f_i = \rho_f g_i,\ \ i = 1,\, 2,\, 3.$$

13.6. Linear Isotropic Theory of Poroelasticity.

Togetthen with constitutive laws, mass and momentum conservation equations need to be introduced in order to construct a well posed mathematical system for the description of the stress, pore pressure, flux and displacement in the medium, concluding in a system of field equations, which include Darcy's law.

Since only four constants can be independently selected, one constant for the deviatoric response, and three for the volumetric ones, for the presentation of the linear theory, the drained and undrained Poisson ratios, ν and ν_u are adopted, to end up with $\{G,\, \alpha,\, \nu,\, \nu_u\}$ as the fundamental set of constants, where

$$\left\{\begin{array}{lcl} \nu & = & \dfrac{3K - 2G}{2(3K + G)} \\ \\ \nu_u & = & \dfrac{3K_u - 2G}{2(3K_u + G)} \end{array}\right\}$$

13.7. Three Other Parameters Also Play Pivotal Roles.

$$\left\{\begin{array}{lcll} B & = & \dfrac{3(\nu_u - \nu)}{\alpha(1 - 2\nu)(1 + \nu_u)}, & B \leq 1 \\ \\ M & = & \dfrac{2G(\nu_u - \nu)}{\alpha^2(1 - 2\nu_u)(1 - 2\nu)}, & M < \infty \\ \\ \eta & = & \dfrac{\alpha(1 - 2\nu)}{2(1 - \nu)}, & \eta \leq 0.5 \end{array}\right.$$

The range of α was [0, 1] and for ν is [0, 0.5], and the values of α and ν_u control the magnitude of the poroelastic effects.

The local stress balance under consideration of static equilibrium leads to:

$$\sum_{j=1}^{3} \frac{\partial \mathcal{T}_{ij}}{\partial x_j} = -F_i, \quad i = 1,\, 2,\, 3.$$

where $F_i = \rho g_i$ is the body force component per unit volume of the bulk material, $\rho = (1-\phi)\rho_s + \phi\rho_f$ is the bulk density, ρ_s and ρ_f are the densities of the solid and the fluid phase respectively.

The continuity equation for the fluid phase is:

$$\frac{\partial \zeta}{\partial t} + \mathrm{div} q = \gamma,$$

where γ is the source density (the rate of injected fluid volume per unit volume of the porous solid), and $q = (q_1,\, q_2,\, q_3)$ is the fluid velocity field, the fluid density variation effect being ignored for this linearized version.

13.8. The 9 Field Equations.

Linear isotropic poroelastic processes are finally described by:

1.- The constitutive equations for the porous solid, in one of the various forms (25)-(28), which are obtained from previous equation (22), but now in terms of $\{G,\, \alpha,\, \nu,\, \nu_u\}$:

i) Selecting the pore pressure p as the coupling term yields: strain–stress,

$$2G\varepsilon_{ij} = \mathcal{T}_{ij} - \frac{\nu}{(1+\nu)}\big(\mathcal{T}_{11} + \mathcal{T}_{22} + \mathcal{T}_{33}\big)\delta_{ij} + \frac{\alpha(1-2\nu)}{(1+\nu)} p\delta_{ij} \tag{25}$$

or the stress-strain equation,

$$\tau_{ij} + \alpha p \delta_{ij} = 2G\varepsilon_{ij} + \frac{2G\nu}{(1-2\nu)}\varepsilon\delta_{ij} \tag{26}$$

(recall that $\varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$)

ii) And if ζ is adopted as the coupling term,

$$2G\left(\varepsilon_{ij} - \tfrac{1}{3}B\zeta\delta_{ij}\right) = \mathcal{T}_{ij} - \frac{\nu_u}{1+\nu_u}\big(\mathcal{T}_{11} + \mathcal{T}_{22} + \mathcal{T}_{33}\big)\delta_{ij} \tag{27}$$

with

$$B = \frac{3(\nu_u - \nu)}{\{\alpha(1-2\nu)(1+\nu_u)\}}.$$

$$\mathcal{T}_{ij} = 2G\varepsilon_{ij} + \frac{2G\nu_u}{(1-2\nu_u)}\varepsilon\delta_{ij} - \alpha M\zeta\delta_{ij} \tag{28}$$

with

$$M = \frac{2G(\nu_u - \nu)}{\alpha^2(1-2\nu_u)(1-\nu)}$$

2.- The response equation for the pore fluid, obtained from (23):

$$2G\zeta = \frac{\alpha(1-2\nu)}{(1+\nu)}\left\{\mathcal{T}_{11} + \mathcal{T}_{22} + \mathcal{T}_{33} + \frac{3}{B}p\right\} \tag{29}$$

$$p = M(\zeta - \alpha\varepsilon) \tag{30}$$

3.- Darcy's law for the fluid velocity field $\vec{q} = (q_1, q_2, q_3)$:

$$q_i = -k\left(\frac{\partial p}{\partial x_i} - f_i\right), \quad \text{where } k = \frac{k^*}{\mu}, \tag{31}$$

with k^* being the intrinsic permeability, having dimensions of length squared, and μ is the fluid viscosity, and $f_i = \rho_f g_i$, g_i being the gravity component in the $i-$direction, ρ_f being the fluid density.

4.- The equilibrium equations

$$\sum_{j=1}^{3} \frac{\partial \mathcal{T}_{ij}}{\partial x_j} = -F_i, \quad i = 1, 2, 3, \tag{32}$$

with $F_i = \rho g_i$; and ρ is the bulk density.

5.- Continuity equation for the fluid phase.

$$\frac{\partial \zeta}{\partial t} + \text{div}\vec{q} = \gamma, \tag{33}$$

where γ is the source density.

6.- Compatibility equations.

In order to ensure a single valued continuous displacement solution u_i, the strain field must satisfy some compatibility requirements, which are identical to those derived in elasticity:

$$\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \varepsilon_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 \varepsilon_{jl}}{\partial x_i \partial x_k} = 0 \tag{34}$$

From the constitutive and the equilibrium (32) equations, the Beltrami–Michell compatibility equations for poroelasticity are:

$$\nabla^2 \mathcal{T}_{ij} + \frac{1}{(1+\nu)}\frac{\partial^2}{\partial x_i \partial x_j}\left(\sum_{k=1}^{3}\mathcal{T}_{kk}\right) + 2\eta\left(\delta_{ij}\nabla^2 p + \frac{1-\nu}{1+\nu}\frac{\partial^2 p}{\partial x_i \partial x_j}\right) = -\frac{\nu}{1-\nu}\delta_{ij}\mathrm{div}F - \left(\frac{\partial F_i}{\partial x_j} + \frac{\partial F_j}{\partial x_i}\right) \tag{35}$$

contracting (35), one obtains a useful relation:

$$\nabla^2(\mathcal{T}_{11} + \mathcal{T}_{22} + \mathcal{T}_{33} + 4\eta p) = -\frac{1+\nu}{1-\nu}\mathrm{div}F \tag{36}$$

For plane strain, this equation reduces to:

$$\nabla^2(\mathcal{T}_{11} + \mathcal{T}_{22} + 2\eta p) = -\frac{1}{1-\nu}\left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right) \tag{37}$$

Harmonic relations can be found between p and ε, p and ζ and ε and ζ

$$\nabla^2\left(p - \left(\frac{G}{\eta}\right)\varepsilon\right) = \frac{1}{\alpha}\mathrm{div}F \tag{38}$$

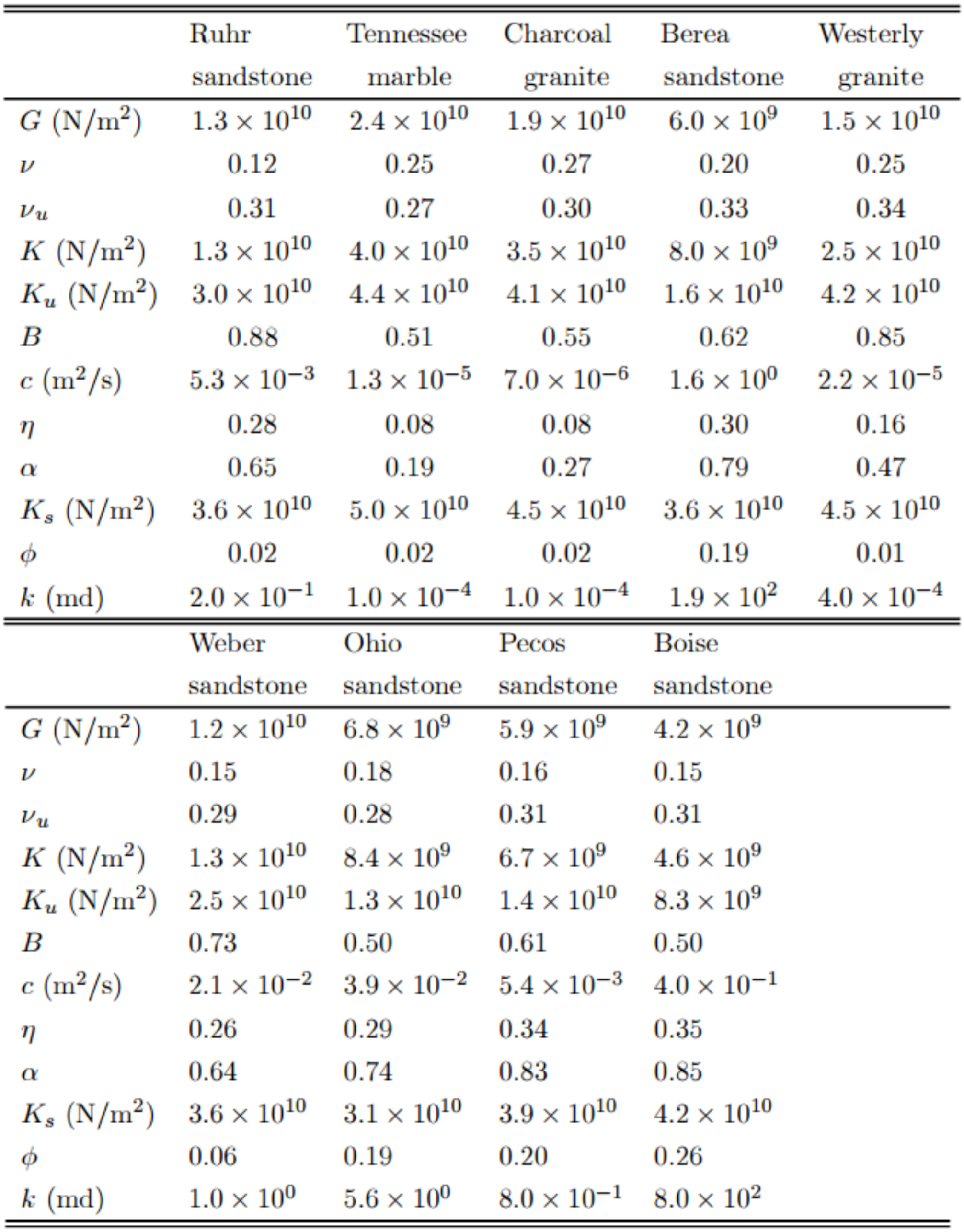
$$\nabla^2(Sp - \zeta) = \frac{\eta}{G}\mathrm{div}F \tag{39}$$

$$\nabla^2\left(\zeta - \frac{GS}{\eta}\varepsilon\right) = \frac{1}{\alpha M}\mathrm{div}F \tag{40}$$

The following conversion of constants is useful while using the basic set $\{G,\ \alpha,\ \nu,\ \nu_u\}$:

$$\begin{aligned}
K &= \frac{2(1+\nu)G}{3(1-2\nu)}; \qquad H' = \frac{K}{\alpha} \\
B &= \frac{\alpha M}{K + \alpha^2 M} = \frac{(Q+R)}{\phi\left[\dfrac{(Q+R)^2}{R} + K\right]} = \frac{R'}{H'} = \frac{\alpha R'}{K} \\
K_u &= K\left(1 + \frac{KR'}{(H')^2 - KR'}\right) = K\left(1 + \frac{KB}{\left(\frac{K}{\alpha}\right) - KB}\right)
\end{aligned}$$

There are tables of poroelastic constants for several rocks:



	Ruhr sandstone	Tennessee marble	Charcoal granite	Berea sandstone	Westerly granite
G (N/m^2)	1.3×10^{10}	2.4×10^{10}	1.9×10^{10}	6.0×10^{9}	1.5×10^{10}
ν	0.12	0.25	0.27	0.20	0.25
ν_u	0.31	0.27	0.30	0.33	0.34
K (N/m^2)	1.3×10^{10}	4.0×10^{10}	3.5×10^{10}	8.0×10^{9}	2.5×10^{10}
K_u (N/m^2)	3.0×10^{10}	4.4×10^{10}	4.1×10^{10}	1.6×10^{10}	4.2×10^{10}
B	0.88	0.51	0.55	0.62	0.85
c (m^2/s)	5.3×10^{-3}	1.3×10^{-5}	7.0×10^{-6}	1.6×10^{0}	2.2×10^{-5}
η	0.28	0.08	0.08	0.30	0.16
α	0.65	0.19	0.27	0.79	0.47
K_s (N/m^2)	3.6×10^{10}	5.0×10^{10}	4.5×10^{10}	3.6×10^{10}	4.5×10^{10}
ϕ	0.02	0.02	0.02	0.19	0.01
k (md)	2.0×10^{-1}	1.0×10^{-4}	1.0×10^{-4}	1.9×10^{2}	4.0×10^{-4}

	Weber sandstone	Ohio sandstone	Pecos sandstone	Boise sandstone
G (N/m^2)	1.2×10^{10}	6.8×10^{9}	5.9×10^{9}	4.2×10^{9}
ν	0.15	0.18	0.16	0.15
ν_u	0.29	0.28	0.31	0.31
K (N/m^2)	1.3×10^{10}	8.4×10^{9}	6.7×10^{9}	4.6×10^{9}
K_u (N/m^2)	2.5×10^{10}	1.3×10^{10}	1.4×10^{10}	8.3×10^{9}
B	0.73	0.50	0.61	0.50
c (m^2/s)	2.1×10^{-2}	3.9×10^{-2}	5.4×10^{-3}	4.0×10^{-1}
η	0.26	0.29	0.34	0.35
α	0.64	0.74	0.83	0.85
K_s (N/m^2)	3.6×10^{10}	3.1×10^{10}	3.9×10^{10}	4.2×10^{10}
ϕ	0.06	0.19	0.20	0.26
k (md)	1.0×10^{0}	5.6×10^{0}	8.0×10^{-1}	8.0×10^{2}

Table 1: Poroelastic Constants for Various Materials.
Source: Detourmay and Cheng (1993), [9], p. 25.

In reference [11], Biot and introduced constants Q, R and M, and the conversion to the present set of constants can be done with

$$\begin{aligned} R &= \frac{\phi^2(K_u - K)}{\alpha^2} \\ Q &= \frac{\phi(\alpha - \phi)(K_u - K)}{\alpha^2} \\ M &= \frac{R}{\phi^2} = \frac{(K_u - K)}{\alpha^2}, \qquad \text{or} \qquad K_u = K + \alpha^2 M \end{aligned}$$

In order to obtain useful solution algorithms, a reduced number of variables is used.

One scheme uses as reduced variables u_i and p, and then the field equations consist of a Navier –type equation for u_i and a diffusion equation for p (both containing a coupling term).

The other approach is based on using u_i and ζ as reduced variables, with a Navier–type equation for u_i and a diffusion equation for ζ this time, but in this second approach, **the diffusion equation is uncoupled from the Navier–type equation** (p, alone).

13.9. Navier Equations.

A Navier–type equation for the displacement u_i is obtained by substituting into the equilibrium equation (32), the constitutive relations (26) or (28), with ε_{ij} expressed in terms of the displacement gradient using

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Two forms of the Navier equation exist, depending on which constitutive relation, (26) or (28), is used:

$$G\nabla^2 u_i + \frac{G}{(1-2\nu)} u_k, k_i = \alpha \frac{\partial p}{\partial x_i} - F_i \tag{41}$$

$$G\nabla^2 u_i + \frac{G}{1-2\nu_u} u_k, k_i = \alpha M \frac{\partial \zeta}{\partial x_i} - F_i \tag{42}$$

where

$$u_{k,k_i} = \sum_{k=1}^{3} \frac{\partial^2 u_k}{\partial x_k \partial x_i} = \frac{\partial^2 u_1}{\partial x_1 \partial x_i} + \frac{\partial^2 u_2}{\partial x_2 \partial x_i} + \frac{\partial^2 u_3}{\partial x_3 \partial x_i}$$

In (41), the coupling term α grad p, and in (42) the coupling term αM grad ζ, can be viewed as some "body force".

13.10. Diffusion Equations.

Two diffusion equations are derived, one for p, and the other for ζ.

I) Consider first the diffusion equation for p, that will uncouple from the Navier equation.

Combination of Darcy's law (31), the continuity equation (33), and the constitutive relation (30) yields

$$\frac{\partial p}{\partial t} - kM\nabla^2 p = -\alpha M \frac{\partial \varepsilon}{\partial t} + M\left(\gamma - k\left[\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right]\right) \tag{43}$$

The diffusion of pore pressure is thus coupled with the rate of change of the volumetric strain. Under steady – state conditions equation (43) certainly uncouples and becomes a Poisson equation. (43) can also uncouple at all time under certain circumstances.

II) The diffusion equation for ζ is deduced from (31) and (33), but taking into account the relationship

$$\nabla^2 (Sp - \zeta) = \frac{\eta}{G}\mathrm{div}F,$$

where the parameter

$$S = \frac{(1-\nu_u)(1-2\nu)}{M(1-\nu)(1-2\nu_u)},$$

represents a "storage coefficient".

The parameter S is defined under the particular conditions of uniaxial strain for ζ and constant normal stress in the direction of the strain (or zero volumetric strain for p).

Now, the diffusion equation has the form:

$$\frac{\partial\zeta}{\partial t} - c\nabla^2\zeta = \frac{\eta c}{G}\mathrm{div}F + \gamma - k\mathrm{divf} \tag{44}$$

where the diffusitivy coefficient c is given by:

$$c = \frac{k}{S} = \frac{2kG(1-\nu)(\nu_u-\nu)}{\alpha^2(1-2\nu)^2(1-\nu_u)}$$

The coefficient "c" is also sometimes called the generalized consolidation coefficient (see [12]), because it is identical to the Terzaghi consolidation coefficient under one–dimensional consolidation.

The diffusion equation (44) for ζ is thus uncoupled at all times, contrary to the diffusion equation(43) for p.

13.11. Solution of Boundary–Value Problems.

To ensure the existence and the uniqueness of the mathematical solution for a system of PDE's describing the response of a poroelastic material, a set of "well–posed" initial and boundary conditions is needed.

For a poroelastic medium, boundary conditions are required for both the porous solid and the fluid. Dirichlet type conditions consist in prescribing the solid displacement u_i and the pore pressure p, while Neumann type conditions correspond to imposing the traction

$$t_i = \sum_{j=1}^{3} \mathcal{T}_{ij} n_j \qquad \text{and the normal flux} \qquad \langle \vec{q}, \hat{\boldsymbol{n}} \rangle = \sum_{i}^{3} q_i n_i.$$

Hydraulic Fracturing involves several strongly coupled processes: Fracture opening, viscous fluid flow in the fracture, diffusion of fracturing fluid in the porous formation, and propagation of the fracture. In this report, only the relationship between **the fracture width** and **the fluid pressure in the fracture** will be addressed in two steps: First, the response of a stationary fracture embedded in an infinite, two dimensional, poroelastic medium will be studied. Next, the response of a vertical hydraulic fracture bounded by impermeable elastic layers is analyzed.

13.12. Griffith Crack Problem.

Given a crack with length L and internally loaded by a fluid at pressure p^*, consider the problem of calculating **the time evolution of the average fracture width** "W".

The fluid pressure loading will decomposed into **mode 1** and **mode 2**. Defining the dimensionless time $\mathcal{T} = 4ct/(L^2)$, the poroelastic response of the fracture to a step loading will be expressed in terms of two functions $W^{(1)}(\mathcal{T})$ and $W^{(2)}(\mathcal{T})$.

13.13. General Preliminaries.

The so called "**Fundamental Problems**" **involve simple geometries**, where at leat one of the boundaries is subject to a prescribed constant normal stress and/or pore pressure suddenly applied at $t = 0^+$, so that letting $H(t)$ denote Heaviside's unit step function, it is convenient to consider **two fundamental loading modes:**

A) **Mode 1**, With $\mathcal{T}_n^{(1)} = -p^* H(t)$ and $p^{(1)} = 0$.

B) **Mode 2**, with $\mathcal{T}_n^{(2)} = 0$ and $p^{(2)} = p^* H(t)$.

Here, the superscripts (1) and (2) are used to designate the corresponding mode, and later these two loadings can be superposed to match any boundary conditions where pore pressure p and normal stress $\mathcal{T}_n$ are arbitrarily imposed.

To get a preliminary idea about the way the governing equations are set up, consider first **uniaxial strain problems**. By definition, only one non–zero normal strain, ε_{xx} say, is considered, and all field quantities vary only along x–direction. Thus, under uniaxial strain condition, equations (25), (26), (27) and (28) become:

$$\mathcal{T}_{xx} = \frac{2G(1-\nu)}{1-2\nu}\varepsilon_{xx} - \alpha p \tag{45}$$

$$\mathcal{T}_{yy} = \mathcal{T}_{zz} = \frac{\nu}{1-\nu}\mathcal{T}_{xx} - 2\eta p \tag{46}$$

$$\mathcal{T}_{xx} = \frac{2G(1-\nu_0)}{1-2\nu_u}\varepsilon_{xx} - \alpha M\zeta \tag{47}$$

$$\mathcal{T}_{yy} = \mathcal{T}_{zz} = \frac{\nu_u}{1-\nu_u}\mathcal{T}_{xx} - 2\eta M\zeta \tag{48}$$

depending on whether p or ζ is chosen as the coupling term under undrained conditions, the pore pressure is proportional to $\mathcal{T}_{xx}$:

$$p = -\frac{B(1+\nu_u)}{3(1-\nu_u)}\mathcal{T}_{xx} = -\frac{\eta}{GS}\mathcal{T}_{xx}$$

There is only one non–trivial equilibrium equations (32), which shows that, in the absence of body force, $\mathcal{T}_{xx}$ is independent of x, although it can still be a function of time. Expressing $\varepsilon_{xx} = \frac{\partial}{\partial x}u_x$, **one obtains an equation for $\boldsymbol{u_x}$** (the Navier equation), using (45) or (47)

$$\frac{2G(1-\nu)}{1-2\nu}\frac{\partial^2 u_x}{\partial x^2} - \alpha\frac{\partial p}{\partial x} = 0.$$

And, expressing ε_{xx} in terms of p and $\mathcal{T}_{xx}$, using (45) **one simplifies the diffusion equation for the pore pressure to:**

$$\frac{\partial p}{\partial t} - c\frac{\partial^2 p}{\partial x^2} = \frac{\eta}{GS}\frac{d\mathcal{T}_{xx}}{dt},$$

and if $\mathcal{T}_{xx}$ is a specified stress condition, we have an inhomogeneous diffusion equation with known right hand side, and p can be solved independently of the displacement u_x in this case.

Besides uniaxial strain problems, one can considerer also **plane strain conditions** while **working with a cylindrical geometry** under conditions of planar deformation and axisymmetry, the displacement field **is obviously irrotational**, and characterized by the only non–zero component $u_r(r,\, t)$, and the pore pressure diffusion equation takes the particular form:

$$\frac{\partial p}{\partial t} - c\left(\frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial r^2}\right) = -\eta\frac{1-\nu}{GS}\frac{d}{dt}\big(\mathcal{T}_{rr} + \mathcal{T}_{\theta\theta} + 2\eta p\big)$$

Under Plane strain, we have

$$\mathcal{T}_{33} = \nu\big[\mathcal{T}_{11} + \mathcal{T}_{22}\big] - \alpha(1-2\nu)p, \qquad \varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$$

13.14. Early Time Evolution of Stress near a Permeable Elastic Boundary.

Stress concentrations at the permeable boundaries present in some simple geometries, constitute a generic process taking place near the permeable boundary of a poroelastic domain, and an analysis of this near–boundary process will be outlined below, followed by a discussion of **the problem of rate–effects in the tensile failure of fluid–pressurized cavities.**

A) Early time stress concentration.

Consider a poroelastic domain adjacent to **a segment of boundary** Γ. It is assumed that this domain **is under condition of plane strain and pore pressure free.**

Consider a local orthogonal curvilinear system of coordinates (l, n) with the n–axis in the same direction as the outward normal to Γ, and let ρ denote the local radius of curvature of the boundary.

Plane Section of Poroelastic Domain and Γ:

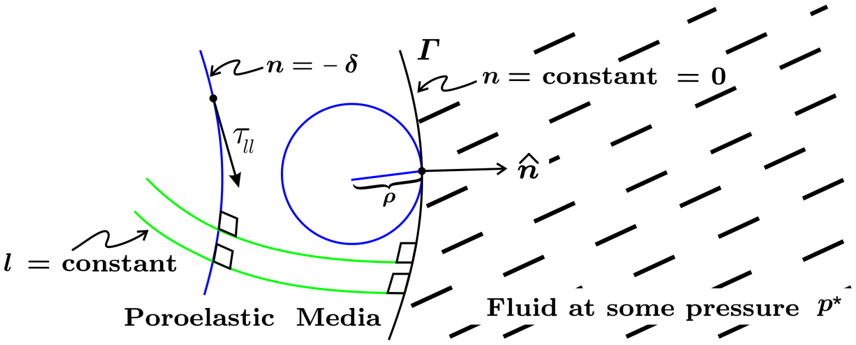


Figure 11. Plane Section of Poroelastic Domain and Γ.

Under the mode 2 loading by means of a sudden contact at time $t = 0^+$ of Γ with a fluid maintained at a constant pressure p^* for $t \geq 0$, then, at early time, the depth of penetration δ of the "diffusion front" is very small compared to the radius of curvature ρ (and to any other relevant length characterizing the problem) i.e., $\frac{\delta}{\rho} \ll 1$, and all other dimensions then appear infinite at the length scale δ. Therefore, a one-dimensional solution of the poroelastic equations is expected in the neighborhood of the boundary. The early time evolution of the stress near the boundary is obtained as the following local relationship between the tangential stress and the pore pressure:

$$\tau_{ll} = -2\eta p, \quad for \quad -\delta < \eta < 0$$

Under the mode 1 loading, there will be an induced pore pressure, denoted as p^u, due to the application of mechanical loading on the poroelastic solid and at the boundary, the "undrained" pore pressure is given by

$$p^u = -\frac{B(1+\nu_u)}{3}(\tau_{ll}^u - p^*), \quad \text{with } p^u \text{ harmonic with } \zeta = 0$$

and at early time, $\tau_{ll} - \tau_{ll}^u = -2\eta(p - p^u)$.

It is worth recalling **the plane strain compatibility equation**, obtained after substituting the constitutive expression (26) into the equilibrium equation, and taking the divergence yields:

$$\nabla^2 \left(p - \frac{G}{\eta}\varepsilon\right) = \frac{1}{\alpha}\left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right)$$

The integration of this equation yields:

$$\mathcal{T}_{ll} + \mathcal{T}_{nn} + 2\eta p = f,$$

where the harmonic function f depends on position and time, but only in time if the displacement field is irrotational.

13.15. Griffith Crack with length *L* and Internally Loaded by a Fluid at Pressure p^*, Mode 1 and Mode 2.

Mode 1 Loading.

The instantaneous average fracture aperture at $t = 0^+$ is given by Sneddon's solution [13] with undrained elastic constants,

$$W^{(1)}(0^+) = \frac{\pi L(1-\nu_u)p^*}{4G} \tag{49}$$

while the long term value $W^{(1)}(\infty)$ is given by the same expression with ν_u replaced by ν. The final fracture aperture is greater than the initial one (this is expected since the poroelastic material is softer when drained). The maximum time–dependent width increase, $\Delta W^{(1)}(\infty)$, experienced by the fracture is thus given by

$$\Delta W^{(1)}(\infty) = \frac{\pi L(\nu_u - \nu)p^*}{4G} \tag{50}$$

It is convenient to introduce the dimensionless width response function $f^{(1)}(\mathcal{T})$, which varies between 0 and 1 es $\mathcal{T}$ increases from 0 to ∞.

$$W^{(1)}(\mathcal{T}) = W^{(1)}(0^+) + \Delta W^{(1)}(\infty) f^{(1)}(\mathcal{T}) \tag{51}$$

The response function can be computed by modelling the fracture using a distribution of singularities to account for the discontinuity of flux and displacement that characterizes the crack [14]. A fracture in a poroelastic medium is a surface across which the solid displacement and the normal fluid flux are generally discontinuos. Such a discontinuity surface can mathematically be simulated by a distribution over time and space of impulse point displacement discontinuities and sources.

The function $f^{(1)}(\mathcal{T})$ depends on the value of ν, ν_u and α.

Mode 2 Loading.

When a constant pore pressure p^* is applied on the fracture faces, the crack width decreases from 0 to the asymptotic value

$$W^{(2)}(\infty) = -\frac{-\pi(1-\nu)L\eta p^*}{4G} \tag{52}$$

At large time, the pore pressure diffusion equation (43) uncouples from the volumetric strain rate (to eventually degenerate into the Laplace equation) and, as a consequence, the induced displacement field becomes irrotational.

In these circumstances, the relationship between this stress and the pore pressure p in given by

$$\mathcal{T}_{xx} + \mathcal{T}_{yy} = -2\eta p \tag{53}$$

As $t \longrightarrow \infty$, the pore pressure p in the region around the fracture approaches the asymptotic value of p^*, and induces, by symmetry, a uniform confining stress $\mathcal{T}_{xx} = \mathcal{T}_{yy}$.

It then follows from (53) that

$$\mathcal{T}_{xx} = \mathcal{T}_{yy} = -\eta p^* \quad \text{as} \quad t \longrightarrow \infty \tag{54}$$

A compressive stress of magnitude ηp^*, sometimes called "back–stress", is thus generated across the fracture path at $\mathcal{T} \longrightarrow \infty$ by mode 2 loading.

To obtain the solution of a fracture in a drained medium with zero stresses on the surface, one must superpose the solution of a Griffith fracture in a drained medium with a tensile stress $\mathcal{T}_n = \eta p^*$ applied on the fracture faces.

Now the asymptotic value of the fracture width is hence given by (52), or equivalently by

$$W^{(2)}(\infty) = -\eta W^{(1)}(\infty) \tag{55}$$

The dimensionless response function

$$f^{(2)}(\mathcal{T}) = \frac{W^{(2)}(\mathcal{T})}{W^{2}(\infty)}$$

be calculated in the same manner as $f^{(1)}(\mathcal{T})$ (see [14])

14. A Note on Uniqueness of Solutions and High–Order Accurate Mimetic Finite–Difference Schemes.

It should be noted that in a finite domain problem, to give exclusively Neumann–type conditions, namely, tractions and fluid flux, allows no–uniqueness to appear. The solution is then defined only within some arbitrary rigid body motions and constant fluid pressure.

Also, these boundary conditions can be alternated to form **a mixed Boundary Value Problem of Robin Type.**

As for the initial conditions, one needs either an initial stress field or displacement field, and a pressure field or a flux field. The conditions specified must themselves satisfy some constraints, such as the equilibrium equation and the "compatibility equations" discussed in subsection 13.8 item 6, p. 44.

If the initial conditions are in an equilibrated state, namely satisfying the governing equations in steady state, they can simply be ignored, as we need then only to solve the perturbed state.

The high–order accurate mimetic finite difference method using the Castillo–Grone operators, "A High Order Stable Mimetic Finite–Difference Scheme for 3D-Acoustic wave motions", submitted by Castillo, J. E. and Miranda, G. F., could be used as an analogical guide for the treatment of Robin–type Boundary Conditions in a finite 3D-domain, with the adequate replacement of the time–stepping needed for diffusion problems instead of wave motion evolution.

Some additional considerations upon hydraulic fractures ca be consulted in section 6.5. of [9].

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Larry Mendoza is a Venezuelan System Engineer who is currently undertaking graduate studies at Computation School of Universidad Central de Venezuela, and is becoming familiar with some of the applications that are of interest at CSRC.

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