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Abstract

In this paper, High-Order compact Castillo-Grone's mimetic divergence and gradient operators are introduced. The numerical tests demonstrate the effectiveness of the compact mimetic implementation.

Keywords: Castillo-Grone's Mimetic Operators, High-Order Operators, Compact Scheme

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1. Introduction

Without any doubt a numerical model must incorporate proper physics in order to deliver acceptable numerical results. However, that is not all. The numerical scheme that is used to solve the PDEs, also have major impact on the quality of the results [1]. Different schemes have different requirements. Both the accuracy and the performance of a model will vary based on the discretization scheme in use.

The majority of the equations, describing physical phenomena, are written using the gradient, divergence, and curl operator. These operators, depending on the field they are applied to, have a physical meaning. Mimetic discretization method, as their name implies, mimics the physical property of their operator and satisfies them exactly in the discrete environment. Castillo and Grone have developed a set of mimetic operators known as Castillo-Grone's mimetic (CGM) operators. CGM operators have been used in many fields, such as wave propagation, seismic studies, electrodynamics, and image processing. In all fields, they have often outperformed the common approaches used in the respective field[2, 3, 4, 5].

CGM Operators have high accuracies even at the boundaries. The accuracy of the solution in many PDE problems is affected by the accuracy used at the boundaries.

This issue is even more visible in the numerical solution of the Navier-Stokes' equations, so that some times it is said that the solution to NS equations is dominated by the Boundary conditions [6, 1].

Other than being accurate, the implementation of CGM operators is very easy. It looks very much like Finite Differences Schemes (FDS). At first glance, one might even think that CGM is the same as FDS with just different coefficients; however, they have completely different properties and behavior.

High order CGM difference operators are available, currently up to 8th order. However, like any other method, the stencil of the scheme keeps getting larger as the order of accuracy is increased. One solution to increase the order of accuracy without increasing the size of the stencil is to implement the operators in a compact way [7, 8]. There are two different approaches to implement compact schemes which are explained. Both methods estimate the derivative as follows:

$$\left(\frac{\partial u}{\partial x}\right) = M_n u, \tag{1}$$

where M_n is a matrix representing a n-th order accurate discretization scheme and u is the vector of values.

1.1. Compact Scheme: Implicit Approach

In this approach the derivative is calculated as follows:

$$R_n \left(\frac{\partial u}{\partial x}\right) = D_2 u, \tag{2}$$

where D_2 is a matrix representing a second order scheme and R_n is a banded matrix. Equation 2 provides a system of linear equations. Once this system of linear equations is solved $\partial u / \partial x$ is calculated with n^{th} order of accuracy. Hence, $M_n = R_n^{-1} D_2$.

1.2. Compact Scheme: Explicit Approach Approach

The implicit approach requires solving a system of linear equations. In the explicit approach the high order accurate derivative is calculates as follows:

$$\left(\frac{\partial u}{\partial x}\right) = R_n D_2 u, \tag{3}$$

This approach eliminates the need for solving a system of linear equations and here $M_n = R_n D_2$. This paper presents the corresponding R matrices for the Castillo-Grone's Mimetic (CGM) divergence and gradient operator. For example, to calculate the gradient using 8-th order accurate CGM divergence operator, one can write:

$$\left(\frac{\partial u}{\partial x}\right) = D_8 u = R_8^D D_2 u, \tag{4}$$

where D_2 is the second order divergence operator and R_8^D is a matrix such that $D_8 = R_8^D D_2$. The second order accurate CGM gradient is defined as follows:

$$G_2 = \frac{1}{h}\begin{bmatrix} \frac{-8}{3} & 3 & \frac{-1}{3} & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & \frac{1}{3} & -3 & \frac{8}{3} \end{bmatrix}. \tag{5}$$

Likewise, the second order accurate CGM divergence operator is:

$$D_2 = \frac{1}{h}\begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix}, \tag{6}$$

where h is the grid spacing.

2. *R* Matrices corresponding to the Higher order operators

In this section different R matrices for second order gradient and divergence operators are introduced.

$$R_4^G = \frac{1}{h}\begin{bmatrix}
\frac{17958}{14245} & \frac{-8776}{14245} & \frac{154787}{341880} & \frac{-3415}{34188} & \frac{25}{9768} & & \\
\frac{-2}{35} & \frac{941}{840} & \frac{-29}{420} & \frac{1}{168} & & & \\
 & \frac{-1}{24} & \frac{13}{12} & \frac{-1}{24} & & & \\
 & & \frac{-1}{24} & \frac{13}{12} & \frac{-1}{24} & & \\
 & & & \vdots & & & \\
 & & & \frac{-1}{24} & \frac{13}{12} & \frac{-1}{24} & \\
 & & & \frac{1}{168} & \frac{-29}{420} & \frac{941}{840} & \frac{-2}{35} \\
 & & \frac{25}{9768} & \frac{-3415}{34188} & \frac{154787}{341880} & \frac{-8776}{14245} & \frac{17958}{14245}
\end{bmatrix} \tag{7}$$

$$R_6^G = \frac{1}{h} \begin{bmatrix} \frac{71069648}{50278305} & \frac{-7798510061}{6435623040} & \frac{27078841571}{19306869120} & \frac{-17027528737}{19306869120} & \frac{468214927}{1379062080} & \frac{-60604897}{919374720} & \frac{827}{193920} & \frac{-6207}{27859840} & \\ \frac{-62}{1155} & \frac{54511}{49280} & \frac{-1003}{22176} & \frac{-3331}{221760} & \frac{151}{15840} & \frac{-37}{21120} & & & \\ \frac{3}{385} & \frac{-10327}{147840} & \frac{41501}{36960} & \frac{-5009}{73920} & \frac{13}{1760} & \frac{-3}{7040} & & & \\ & \frac{3}{640} & \frac{-29}{480} & \frac{1067}{960} & \frac{-29}{480} & \frac{3}{640} & & & \\ & & \frac{3}{640} & \frac{-29}{480} & \frac{1067}{960} & \frac{-29}{480} & \frac{3}{640} & & \\ & & & \vdots & & & & & \\ & & & \frac{3}{640} & \frac{-29}{480} & \frac{1067}{960} & \frac{-29}{480} & \frac{3}{640} & \\ & & & \frac{-3}{7040} & \frac{13}{1760} & \frac{-5009}{73920} & \frac{41501}{36960} & \frac{-10327}{147840} & \frac{3}{385} \\ & & & \frac{-37}{21120} & \frac{151}{15840} & \frac{-3331}{221760} & \frac{-1003}{22176} & \frac{54511}{49280} & \frac{-62}{1155} \\ & \frac{-6207}{27859840} & \frac{827}{193920} & \frac{-60604897}{919374720} & \frac{468214927}{1379062080} & \frac{-17027528737}{19306869120} & \frac{27078841571}{19306869120} & \frac{-7798510061}{6435623040} & \frac{71069648}{50278305} \end{bmatrix} \quad (8)$$

$$R_8^G = \frac{1}{h}\begin{bmatrix}
\frac{1872}{1231} & \frac{-1261}{716} & \frac{5040}{1849} & \frac{-3287}{1182} & \frac{3594}{1759} & \frac{-853}{843} & \frac{555}{1762} & \frac{-346}{6233} & \frac{125}{24223} & \frac{-15}{32776} & \frac{4}{183113} & & & \\
\frac{-380}{7957} & \frac{1742}{1619} & \frac{149}{5594} & \frac{-1528}{13145} & \frac{83}{851} & \frac{-1043}{21806} & \frac{118}{8863} & \frac{-77}{47381} & & & & & & \\
\frac{35}{5567} & \frac{-316}{5087} & \frac{1600}{1449} & \frac{-105}{2557} & \frac{-83}{5099} & \frac{248}{20229} & \frac{-85}{22524} & \frac{35}{73214} & & & & & & \\
\frac{-4}{3003} & \frac{37}{3317} & \frac{-862}{11545} & \frac{541}{479} & \frac{-277}{3752} & \frac{155}{15099} & \frac{-151}{139776} & \frac{1}{21504} & & & & & & \\
 & \frac{-5}{7168} & \frac{159}{17920} & \frac{-286}{4035} & \frac{1032}{917} & \frac{-286}{4035} & \frac{159}{17920} & \frac{-5}{716} & & & & & & \\
 & & \frac{-5}{7168} & \frac{159}{17920} & \frac{-286}{4035} & \frac{1032}{917} & \frac{-286}{4035} & \frac{159}{17920} & \frac{-5}{716} & & & & & \\
 & & & & & \vdots & & & & & & & & \\
 & & & & & & \frac{-5}{7168} & \frac{159}{17920} & \frac{-286}{4035} & \frac{1032}{917} & \frac{-286}{4035} & \frac{159}{17920} & \frac{-5}{716} & \\
 & & & & & & \frac{1}{21504} & \frac{-151}{139776} & \frac{155}{15099} & \frac{-277}{3752} & \frac{541}{479} & \frac{-862}{11545} & \frac{37}{3317} & \frac{-4}{3003} \\
 & & & & & & \frac{35}{73214} & \frac{-85}{22524} & \frac{248}{20229} & \frac{-83}{5099} & \frac{-105}{2557} & \frac{1600}{1449} & \frac{-316}{5087} & \frac{35}{556} \\
 & & & & & & \frac{-77}{47381} & \frac{118}{8863} & \frac{-1043}{21806} & \frac{83}{851} & \frac{-1528}{13145} & \frac{149}{5594} & \frac{1742}{1619} & \frac{-380}{7957} \\
 & & & \frac{4}{183113} & \frac{-15}{32776} & \frac{125}{24223} & \frac{-346}{6233} & \frac{555}{1762} & \frac{-853}{843} & \frac{3594}{1759} & \frac{-3287}{1182} & \frac{5040}{1849} & \frac{-1261}{716} & \frac{1872}{1231}
\end{bmatrix} \quad (9)$$

2.2. R Matrices for the Divergence Operator

$$R_4^D = \frac{1}{h}\begin{bmatrix} \frac{1045}{1142} & \frac{492}{2291} & \frac{-418}{2371} & \frac{328}{6821} & \frac{-25}{15576} & & & & & \\ \frac{-1}{24} & \frac{13}{12} & \frac{-1}{24} & & & & & & & \\ & \frac{-1}{24} & \frac{13}{12} & \frac{-1}{24} & & & & & & \\ & & & & \vdots & & & & & \\ & & & & & & \frac{-1}{24} & \frac{13}{12} & \frac{-1}{24} & \\ & & & & & \frac{-25}{15576} & \frac{328}{6821} & \frac{-418}{2371} & \frac{492}{2291} & \frac{1045}{1142} \end{bmatrix} \tag{10}$$

$$R_6^D = \frac{1}{h}\begin{bmatrix}
\frac{1239}{1465} & \frac{465}{881} & \frac{-2081}{2823} & \frac{293}{535} & \frac{-237}{1049} & \frac{117}{2648} & \frac{-19}{7819} & \frac{16}{125035} & & & & & \\
\frac{-31}{960} & \frac{582}{559} & \frac{1}{30} & \frac{-21}{320} & \frac{9}{320} & \frac{-3}{640} & & & & & & & \\
\frac{3}{640} & \frac{-29}{480} & \frac{1067}{960} & \frac{-29}{480} & \frac{3}{640} & & & & & & & & \\
 & \frac{3}{640} & \frac{-29}{480} & \frac{1067}{960} & \frac{-29}{480} & \frac{3}{640} & & & & & & & \\
 & & & & & \vdots & & & & & & & \\
 & & & & & & & & \frac{3}{640} & \frac{-29}{480} & \frac{1067}{960} & \frac{-29}{480} & \frac{3}{640} \\
 & & & & & & & \frac{-3}{640} & \frac{9}{320} & \frac{-21}{320} & \frac{1}{30} & \frac{582}{559} & \frac{-31}{960} \\
 & & & & & \frac{16}{125035} & \frac{-19}{7819} & \frac{117}{2648} & \frac{-237}{1049} & \frac{293}{535} & \frac{-2081}{2823} & \frac{465}{881} & \frac{1239}{1465}
\end{bmatrix} \tag{11}$$

$$
R_8^D = \frac{1}{h}\begin{bmatrix}
\frac{397}{501} & \frac{191}{218} & \frac{-730}{427} & \frac{963}{469} & \frac{-1528}{939} & \frac{853}{1020} & \frac{-514}{1955} & \frac{280}{6207} & \frac{-52}{13711} & \frac{16}{47229} & \frac{-4}{245355} & & & & \\
\frac{-269}{10756} & \frac{851}{856} & \frac{218}{1343} & \frac{-483}{1853} & \frac{38}{187} & \frac{-219}{2240} & \frac{149}{5511} & \frac{-59}{17920} & & & & & & & \\
\frac{59}{17920} & \frac{-99}{1928} & \frac{1472}{1355} & \frac{-342}{15509} & \frac{-218}{7221} & \frac{135}{7168} & \frac{-5}{896} & \frac{5}{7168} & & & & & & & \\
\frac{-5}{7168} & \frac{159}{17920} & \frac{-286}{4035} & \frac{1032}{917} & \frac{-286}{4035} & \frac{159}{17920} & \frac{-5}{7168} & & & & & & & & \\
 & \frac{-5}{7168} & \frac{159}{17920} & \frac{-286}{4035} & \frac{1032}{917} & \frac{-286}{4035} & \frac{159}{17920} & \frac{-5}{7168} & & & & & & & \\
 & & & & & & \vdots & & & & & & & & \\
 & & & & & & & & \frac{-5}{7168} & \frac{159}{17920} & \frac{-286}{4035} & \frac{1032}{917} & \frac{-286}{4035} & \frac{159}{17920} & \frac{-5}{7168} \\
 & & & & & & & \frac{5}{7168} & \frac{-5}{896} & \frac{135}{7168} & \frac{-218}{7221} & \frac{-342}{15509} & \frac{1472}{1355} & \frac{-99}{1928} & \frac{59}{17920} \\
 & & & & & & & \frac{-59}{17920} & \frac{149}{5511} & \frac{-219}{2240} & \frac{38}{187} & \frac{-483}{1853} & \frac{218}{1343} & \frac{851}{856} & \frac{-269}{10756} \\
 & & & & \frac{-4}{245355} & \frac{16}{47229} & \frac{-52}{13711} & \frac{280}{6207} & \frac{-514}{1955} & \frac{853}{1020} & \frac{-1528}{939} & \frac{963}{469} & \frac{-730}{427} & \frac{191}{218} & \frac{397}{501}
\end{bmatrix} \tag{12}
$$

3. Tests

To test these matrices, different polynomial functions were selected. The second norm was used to calculate the error as follows:

$$e = ||x - x^*||_2, \tag{13}$$

where x is the numerical results and x^* is the analytic solution. The selected functions are:

$$f_1(x) = x^4, \tag{14}$$

$$f_2(x) = x^6, \tag{15}$$

$$f_3(x) = x^8, \tag{16}$$

$$f_4(x) = x^9 + 30x^7 + 273x^5 + 820x^3 + 576x, \tag{17}$$

$$f_5(x) = sin(x), \tag{18}$$

$$f_6(x) = e^{-0.1x}, \tag{19}$$

and

$$f_7(x) = sin(x)e^x. \tag{20}$$

The errors are reported in table 1.

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Function	Order of Accuracy	D	G
$f_1(x)$	2nd	0.006453	0.009597
	4th	1.1252e-14	8.7793e-15
$f_2(x)$	2nd	0.021088	0.039773
	6th	8.4644e-15	1.5963e-14
$f_3(x)$	2nd	0.04696	0.1013
	8th	1.652e-14	1.7653e-14
$f_4(x)$	2nd	0.2044	0.22598
	4th	1.9303e-5	0.22598
	6th	1.3165e-9	1.4045e-9
	8th	5.0616e-11	6.0166e-6
$f_5(x)$	2nd	7.0537e-5	7.2516e-5
	4th	9.2996e-10	1.0718e-9
	6th	5.1826e-13	5.1722e-13
	8th	5.562e-13	5.457e-13
$f_6(x)$	2nd	7.7885e-8	7.9107e-8
	4th	2.0073e-13	2.2681e-13
	6th	2.0923e-13	2.4114e-13
	8th	2.1493e-13	2.5286e-13
$f_7(x)$	2nd	0.0015063	0.001604
	4th	2.8281e-8	3.8153e-8
	6th	5.2039e-12	6.1804e-12
	8th	3.4476e-12	2.7003e-12

Table 1: Reducing the error using compact CG mimetic operators

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