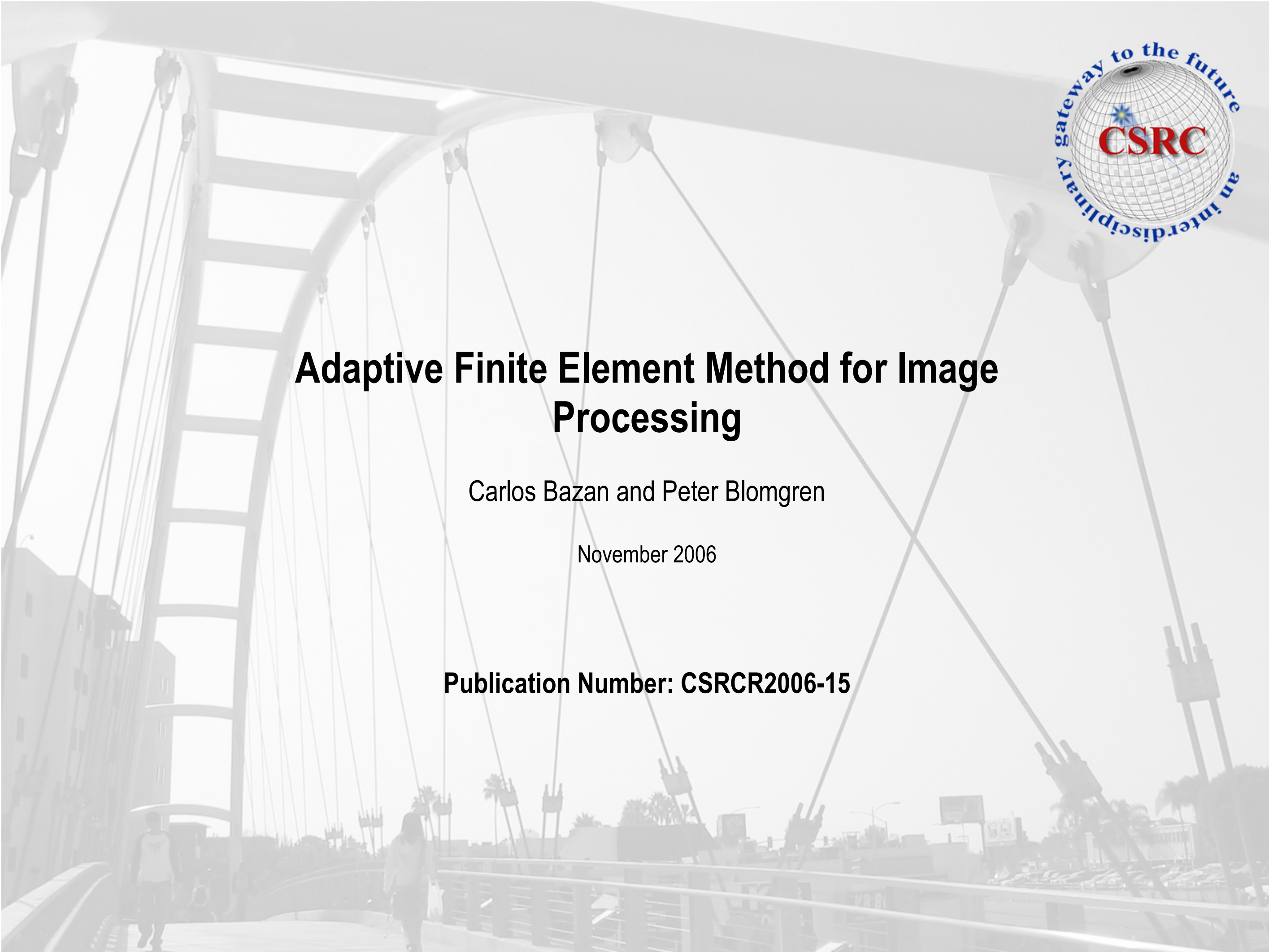

Adaptive Finite Element Method for Image Processing

Carlos Bazan and Peter Blomgren

November 2006

Publication Number: CSRCR2006-15

COMPUTATIONAL SCIENCE & ENGINEERING

SAN DIEGO STATE UNIVERSITY

Computational Science Research Center
College of Sciences
5500 Campanile Drive
San Diego, CA 92182-1245
(619) 594-3430

Carlos Bazan[1] and Peter Blomgren, PhD[2]

Adaptive finite element method for image processing

Abstract An adaptive finite element strategy is employed to solve the Perona-Malik model as modified by Catté, Lions, Morel and Coll for image processing by (often highly) nonlinear diffusion. FEMLAB® and MATLAB® are used to implement the experiments and they prove to be very suitable tools to run this type of problem. Refinement and coarsening of the grids are used as needed and the approach leads to unstructured grids where the efficiency of the remeshing strategy is demonstrated by obtaining very similar results as in the regular grid case, though with fewer unknowns.

Keywords FEMLAB – MATLAB – image processing – adaptive finite element method – parabolic equation – nonlinear diffusion.

1 Physical Background

The diffusion process (mass transfer) is the movement of matter from a high concentration to a low concentration. The equilibrium property is expressed by Fick's law [1]:

$$q = -D \cdot \nabla u \, . \tag{1}$$

The concentration gradient ∇u generates a flux q which attempts to compensate for the gradient. The relation between the gradient ∇u and the flux q is described by a positive definite symmetric matrix D, the diffusion tensor. For the isotropic (homogenous) case, one can replace the diffusion tensor with a scalar-valued − the diffusivity d − that describes the diffusion rate. For the most general anisotropic (inhomogeneous) case, one will have to use the diffusion tensor D.

The phenomenon described above represents the transport of mass without creating or destroying any mass. Therefore, one can state the following continuity equation

$$u_t = -\operatorname{div} q \, . \tag{2}$$

Substituting (1) in (2) yields the diffusion equation

$$u_t = \operatorname{div}\left(D \cdot \nabla u \right) . \tag{3}$$

In the context of image processing the concentration represents the values of the amplitude of the image − gray level intensities or tones of gray. The diffusion tensor D (or diffusivity d) is commonly a function of the concentration u and/or its derivatives, which leads to a nonlinear diffusion process.

2 Nonlinear Diffusion Filtering

During early 1980's the classical transform and filter-based approaches to image processing [12], were replaced by the solving of parabolic PDEs. The work of Alvarez, Guichard, Lions and Morel [13] was fundamental in demonstrating that all scale-spaces that fulfill a few rather natural axioms are governed by parabolic PDEs, with the original image as initial condition. Within these, the parabolic equation proposed by Perona and Malik [2], which incorporate nonlinear diffusion filters, was a milestone in the novel field of image processing.

Many of today's PDE-based image processing and edge detection models are based on the classic Perona-Malik "anisotropic diffusion" method. A few models worth mentioning are curvature-driven equations such as the level set equation of Osher and Sethian [6], nonlinear total variation based models such as the noise removal algorithms of Rudin, Osher and Fatemi [7], active contour model (snakes) of Kass *et al* [15], and anisotropic vector-valued models such as those of Weickert [8] [9], Gerig *et al* [10], and Whitaker [11].

Based on the work of Bänsch and Mikula [3] we seek to find a numerical solution to the Perona-Malik model as modified by Catté, Lions, Morel and Coll [4] using adaptive finite element method.

3 The Diffusion Model

Today's variations on the classic Perona-Malik model stem from the following form

$$u_t - \operatorname{div}\left(g\left(\left| \nabla u \right| \right) \nabla u \right) = 0 \, . \tag{4}$$

1 Computational Science Research Center, San Diego State University, Carlos.Bazan@sdsu.edu
2 Department of Mathematics & Statistics, San Diego State University, Blomgren@terminus.sdsu.edu

We adopt a modified version to this model by Catté- Lions- Morel- Coll:

$$u_t - \operatorname{div}\left(g\left(\left|\nabla G_\tau * u\right|\right)\nabla u\right) = f\left(u_0 - u\right), \qquad (5)$$

defined in the domain Ω with boundary conditions $u_\nu = 0$ on $\partial\Omega$ (where ν is the unit normal vector to the boundary of the domain Ω). The Neumann boundary conditions should guarantee that the filtering does not significantly affect the average grey value of the image. The initial condition is the original image $u(0,\mathbf{x}) = u_0(\mathbf{x})$ in Ω.

In this model $g:\mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing function with $g(0)=1$, $g\left(\sqrt{s}\right)$ is smooth, and we required that $g(s)\to 0$ for $s\to\infty$. $G_\tau \in C^\infty\left(\mathbb{R}^2\right)$ is a smooth kernel, $\int_{\mathbb{R}^2} G_\tau(\mathbf{x})\,d\mathbf{x} = 1$, $\int_{\mathbb{R}^2} \nabla G_\tau d\mathbf{x} \le C_\tau$, $G_\tau(\mathbf{x}) \to \delta_{\mathbf{x}}$ for $\tau \to 0$, where $\delta_{\mathbf{x}}$ is the Dirac delta at point $\mathbf{x}$, $u_0 \in L_\infty(\Omega)$, and the convolution $\nabla G_\tau * u = \int_{\mathbb{R}^2} \nabla G_\tau(\mathbf{x}-\xi)\tilde{u}(\xi)\,d\xi$, where $\tilde{u}$ is a linear and continuous extension of u to $\mathbb{R}^2$.

The diffusion process is governed by the shape of the diffusivity function g, and the gradient ∇u acts as an edge detector. The forcing term $f\left(u_0 - u\right)$ was not in the original Catté- Lions- Morel- Coll model. It was first introduced by Nordström [14] and it forces $u(t,\mathbf{x})$ to remain close to $u_0(\mathbf{x})$, eliminating the need to choose a stopping time.

4 The Diffusion Model

We employ an adaptive meshing scheme to solve a nonlinear stationary problem and use the successive mesh regenerations to provide the scale steps. Refinement and coarsening of the grids are used as needed. The approach leads to unstructured grids by using the efficient mesh regeneration or remeshing technique [5].

The model to solve is (5) where the diffusivity function takes the form $g(s) = 1/\left(1+s^2\right)$ and the forcing term $f\left(u_0-u\right) = -\eta\left(u-u_0\right)$, where η is a constant. For the kernel G_τ we use Gaussian distribution in two dimensions (assuming the mean is zero)

$$G(x,y) = \frac{1}{2\pi\sigma^2} e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}, \qquad (6)$$

where σ is the standard deviation. The kernel has been normalized to avoid the brightness of the image from increasing.

5 Numerical Experiments

This example involves a 256×256 pixel image composed of simple geometric patterns where the original image has been perturbed via the MATLAB function `imnoise` using Gaussian white noise with constant mean and variance (in this example the mean is 0 and the variance is 0.01). Two different cases are presented. In the first case we run the time-dependent model (5) using a regular grid of 65,536 degrees of freedom (DOF). Each DOF corresponds to each pixel and its value represents one grey tone (0 thru 255). In the second case we run the non-linear steady-state model using adaptive grid starting from the regular grid (65,536 DOF).

The original image and the perturbed image are shown in figure 1 and figure 2, respectively. The processed images are shown in figures 3 through 6. The adaptive finite element approach employs only 1 iteration (iteration 0 corresponds to the initial regular grid) to reach the level of diffusion obtained after 47 time-steps with the regular grid (time-dependant model). We use a pixel-wise correlation coefficient to compare the similarity between two images. Other image statistics such as the standard deviation (STD), the mean, and the coefficient of variation (CoV) are also shown below each image.

The computational effort necessary to run the experiments on a computer equipped with a 1.50GHz Intel® Pentium® processor and 1.25 GB of RAM is shown in table 1. The adaptive finite element approach proves to perform very efficiently despite the high nonlinearity of the problem, and very few iterations are needed to reach convergence for each mesh case because of the good initial condition adopted.

Table 1

Experiment Case	CPU time
Regular grid, 47 iteration	285.230 s
Adaptive grid, 1 iteration	116.668 s
Adaptive grid, 2 iterations	153.230 s
Adaptive grid, 5 iterations	274.124 s

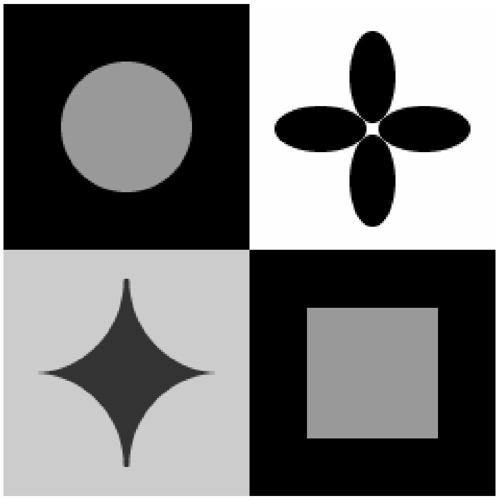

Fig. 1. Original image

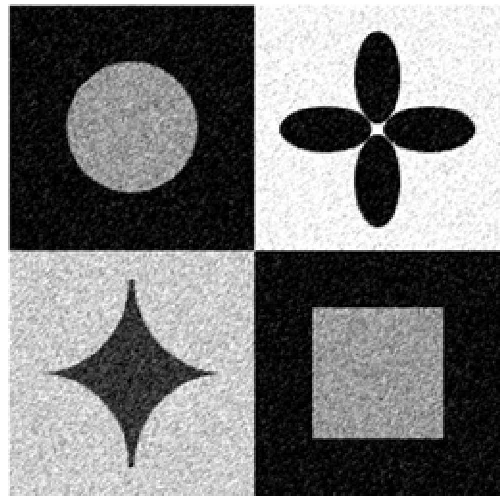

Fig. 2. Noisy image, STD 101.6430,
mean 116.9876, CoV 0.8688, SNR 0.2227

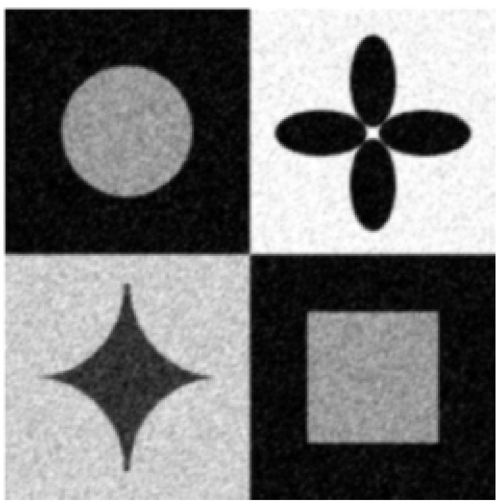

Fig. 3a. Solution after 47 time-steps,
STD 97.9955, mean 116.9821, CoV 1.1860, CPU 285.23 s

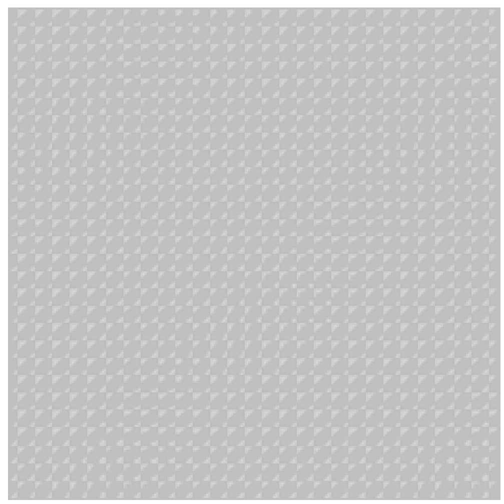

Fig. 3b. Regular mesh with 65,536 DOF

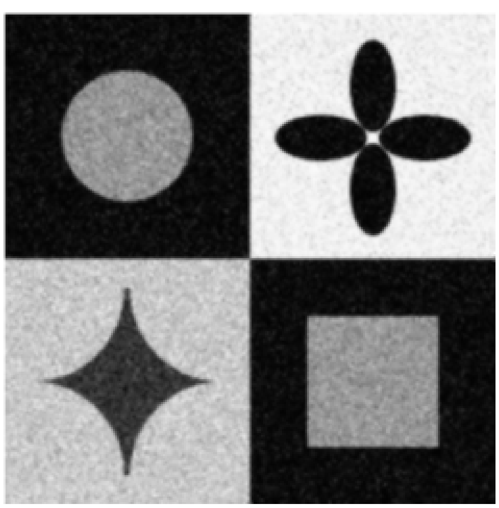

Fig. 4a. Solution after 1 iteration using adaptive mesh,
STD 89.0348, mean 125.2020, CoV 1.4062,
Correlation coefficient 0.99973854209736, CPU 116.668 s

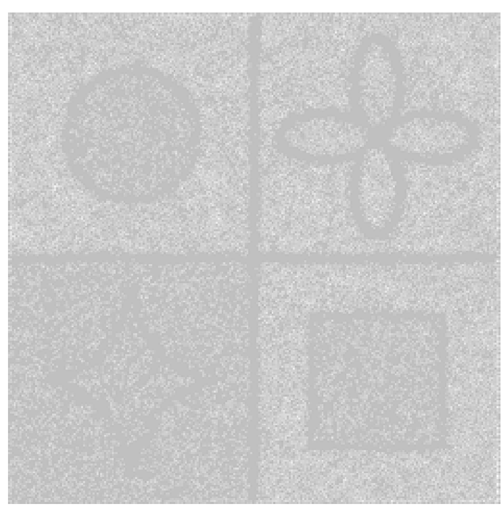

Fig. 4b. Adaptive mesh with 54,488 DOF

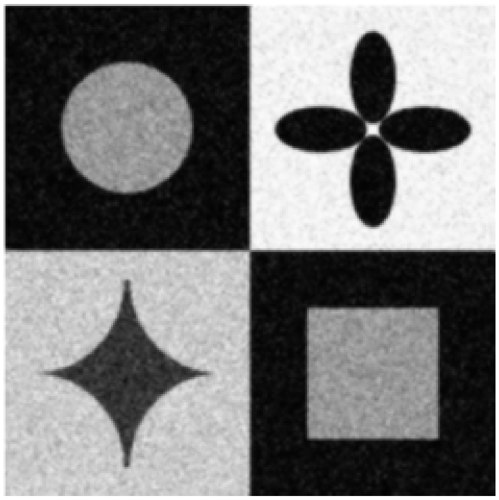

Fig. 5a. Solution after 2 iterations using adaptive mesh,
STD 88.2264, mean 125.7386, CoV 1.4252
Correlation coefficient 0.99972061269544, CPU 153.23 s

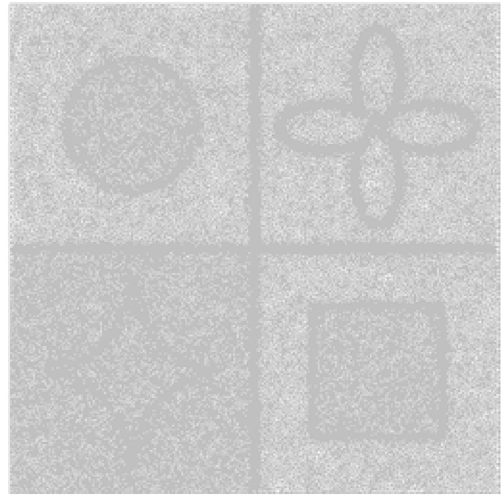

Fig. 5b. Adaptive mesh with 52,072 DOF

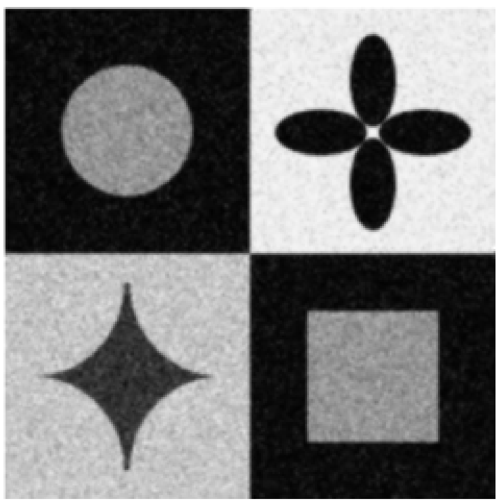

Fig. 6a. Solution after 5 iterations using adaptive mesh,
STD 88.1398, mean 125.6871, CoV 1.4260,
Correlation coefficient 0.99966519090267, CPU 274.124 s

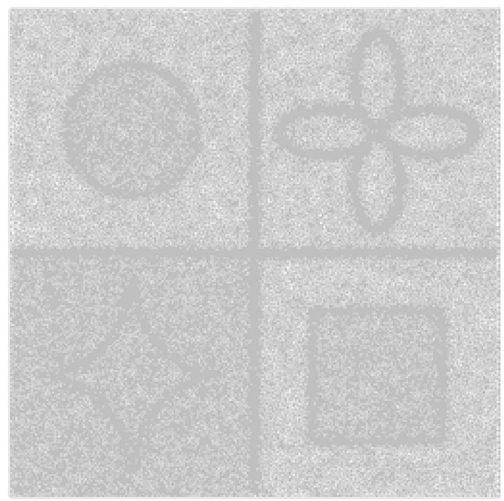

Fig. 6b. Adaptive mesh with 45,696 DOF

Conclusions Adaptive finite element is employed to solve the Perona-Malik model modified by Catté, Lions, Morel and Coll for image processing by nonlinear diffusion. The employment of adaptive grid proved to be a very efficient approach where considerably fewer DOF are necessary to produce similar results to the regular grid case. By using the remeshing approach based on the L^2-norm, nodes are placed following the edges of the image which allows very good edge preservation.

Acknowledgements Thanks Michael Carnohan for helping improve this presentation.

References

[1] J. Weickert, A Review of Nonlinear Diffusion Filtering, *Notes in Computer Science*, Vol. 1252, pp. 3-28, 1997.

[2] P. Perona and J. Malik, Scale Space and Edge Detection Using Anisotropic Diffusion, *Proceedings IEEE Computer Society Workshop on Computer Vision*, 1987.

[3] E. Bänsch and K. Mikula, A Coarsening Finite Element Strategy in Image Selective Smoothing, *Computing and Visualization in Science*, Vol. 1, pp. 53-61, 1997

[4] F. Catté, P.L. Lions, J.M. Morel, and T. Coll, Image Selective Smoothing and Edge Detection by Nonlinear Diffusion, *SIAM Journal of Numerical Analysis*, Vol. 29, pp. 182-193, 1992.

[5] O.C. Zienkiewicz and R. L. Taylor, *The Finite Element Method - The Basis*, Vol. 1, 5th Edition, pp. 402, 2000.
[6] S. Osher and J. Sethian, Front Propagating with Curvature Dependent Speed: algorithms based on the Hamilton-Jacobi formulation, *Journal of Computational Physics*, Vol. 79, pp. 12-49, 1988.
[7] L. Rudin, S. Osher and E. Fatemi, Nonlinear Total Variation Based Noise Removal Algorithms, *Physica D*, Vol. 60, pp. 259-268, 1992.
[8] J. Weickert, Scale-space Properties of Nonlinear Diffusion Filtering with a Diffusion Tensor, *Report No. 110, Laboratory of Technomathematics*, University of Kaiserslautern, P.O. Box 3049, 67653 Kaiserslautern, Germany, 1994.
[9] J.Weickert, Coherence-Enhancing Diffusion of Colour Images, *Proc. Seventh National Symposium on Pattern Recognition and Image Analysis*, VII NSPRIA, Barcelona, April 21-25, 1997.
[10] G. Gerig, O. Kubler, R. Kikinis, F.A. Jolesz, Nonlinear Anisotropic Filtering of MRI Data, *IEEE Trans. Medical Imaging*, Vol. 11, pp. 221-232, 1992.
[11] R. Whitaker, G. Gerig, Vector-Valued Diffusion, *B.M. ter Haar Romeny (Ed.), Geometry-driven diffusion in computer vision*, Kluwer, Dordrecht, pp. 93-134, 1994.
[12] R.C. Gonzalez and R.E. Woods, *Digital image processing*, 2nd Edition, 2001.
[13] L. Alvarez, F. Guichard, P.L. Lions, J.M. Morel, Axioms and Fundamental Equations in Image Processing, *Archive for Rational Mechanics and Analysis*, Vol. 123, pp. 199-257, 1993.
[14] K.N. Nordström, Biased Anisotropic Diffusion: a Unified Regularization and Diffusion Approach to Edge Detection, *Image and Vision Computing*, Vol. 8, Issue 4, pp. 318-327, 1991.
[15] M. Kass, A. Witkin and D. Terzopoulos, Snakes: Active Contour Models, *International Journal of Computer Vision*, Vol. 1, Number 4, pp. 321-331, 1988.