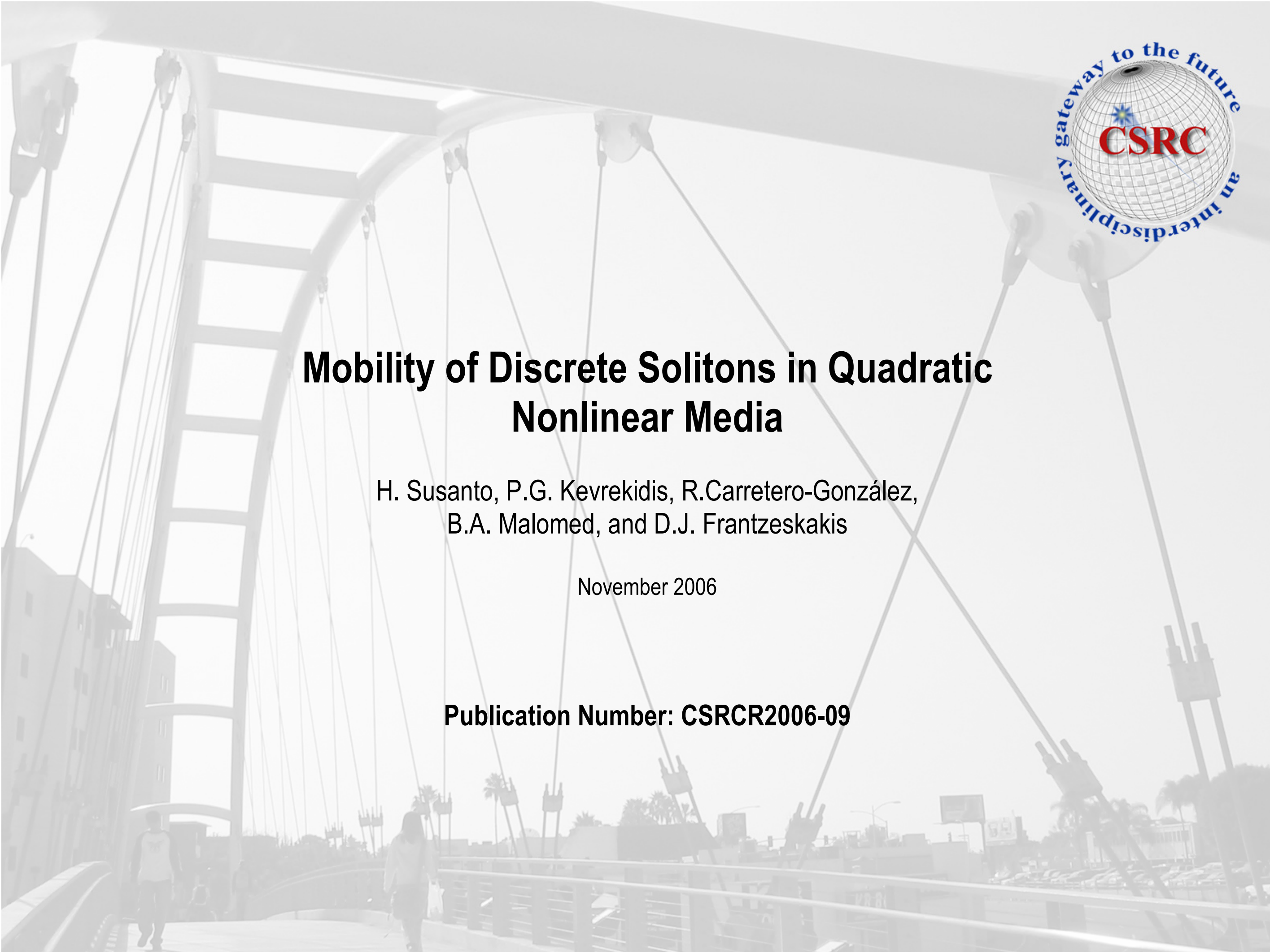

Mobility of Discrete Solitons in Quadratic Nonlinear Media

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Mobility of Discrete Solitons in Quadratic Nonlinear Media

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We study the mobility of solitons in lattices with $\chi^{(2)}$ (second-harmonic-generating) nonlinearity. Contrary to what is known for their $\chi^{(3)}$ counterparts, we demonstrate that discrete quadratic solitons can be mobile not only in the one-dimensional (1D) setting, but also in two dimensions (2D), in *any* direction. We identify parametric regions where an initial kick applied to a soliton leads to three possible outcomes, namely, staying put, persistent motion, or destruction. On the 2D lattice, the solitons survive the largest kick and attain the largest speed along the diagonal direction. The dynamics of the solitons is also studied in an analytical approximation, based on the corresponding Peierls-Nabarro potential.

Introduction. In the past several years, tremendous progress has been made in studies of nonlinear dynamical systems on lattices [1]. To a considerable extent, this development was driven by the continuing expansion of physical applications, such as optical beams in waveguide arrays [2], Bose-Einstein condensates in deep optical lattices [3], transformations of the DNA double strand [4], and so on.

A ubiquitous dynamical-lattice system is represented by the discrete nonlinear Schrödinger equation [1, 2, 5] with cubic ($\chi^{(3)}$) nonlinearity. It has been used to model a variety of experimental settings and demonstrate the formation of discrete solitons, lattice modulational instability, buildup of the Peierls-Nabarro (PN) barrier impeding the motion of solitons, diffraction management, soliton interactions, etc. [6–11].

Substantial activity has also been aimed at lattices with the quadratic ($\chi^{(2)}$) nonlinearity, which was originally introduced to describe the dynamics of Fermi-resonance interface modes in multilayered systems based on organic crystals [12]; more recently, a variety of solutions have been reported in that context [13]. Recently, the interest in $\chi^{(2)}$ lattices was boosted by the experimental realization of discrete $\chi^{(2)}$ solitons in nonlinear optics [14]. A variety of topics have been studied in this context both theoretically and experimentally, including the formation of 1D and 2D solitons [15–17] (see also reviews [18, 19]), modulational instability in periodically-poled LiNb waveguide arrays [20], few-site lattices [21], $\chi^{(2)}$ photonic crystals [22], cavity solitons [23], multi-color localized modes [24], etc.

A fundamental difference of $\chi^{(2)}$ continua from their $\chi^{(3)}$ counterparts [25] is that they feature no collapse in 2D and 3D cases [26], which paves the way to create stable 2D [27] and 3D [28] quadratic solitons. On the other hand, due to the presence of the collapse in 2D and 3D $\chi^{(3)}$ continua, lattice solitons in the corresponding discrete setting are subject to *quasi-collapse*. Thus, they only exist with a norm exceeding a certain threshold value [29], and they are strongly localized (on few lattice sites), hence 2D and 3D $\chi^{(3)}$ solitons are strongly pinned to the lattice and cannot be motile [30].

The absence of the trend to the catastrophic self-compression in the 2D $\chi^{(2)}$ medium suggests that the corresponding lattice solitons may be broad and therefore *mobile*, being loosely bound to the lattice. The aim of this work is to investigate the mobility of 1D and, especially, 2D solitons in $\chi^{(2)}$ lattices. Besides potential applications to photonics, the topic presents fundamental interest, revealing a family of *mobile solitons* in 2D lattices. Thus far, the only example of mobility was provided by solitons in a 2D lattice with saturable nonlinearity [30] (the *Vinetskii-Kukhtarev* model [31], in which the mobility of 1D solitons was examined in Ref. [32]). Experimentally, 1D mobility of strongly anisotropic 2D gap solitons was observed in a (continuous) photorefractive medium with a square photonic lattice [33].

In this work, we identify parametric regions of stable motion of $\chi^{(2)}$ solitons on 1D and 2D lattices and anisotropy of the 2D mobility of the discrete solitons (for motion *off the principal directions* on the lattice). First, we introduce the model and develop an analytical approach to the mobility of solitons based on the analysis of the respective PN barrier. Then, systematic numerical results for the soliton mobility in 1D and 2D lattices are reported.

The model and analytical results. Following Ref. [17], we introduce a system of equations for the fundamental-frequency (FF) and second-harmonic (SH) waves, $\psi_{m,n}(t)$ and $\phi_{m,n}(t)$ on the 2D lattice (its 1D counterpart will be considered too):

$$i\frac{d}{dt}\psi_{m,n} = -\left(C_1\Delta_2\psi_{m,n} + \psi^{\star}_{m,n}\phi_{m,n}\right), \quad (1)$$

$$i\frac{d}{dt}\phi_{m,n} = -\frac{1}{2}\left(C_2\Delta_2\phi_{m,n} + \psi^2_{m,n} + k\phi_{m,n}\right), \quad (2)$$

where $\Delta_2 u_{m,n} \equiv u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1} - 4u_{mn}$, the FF and SH lattice-coupling constants are C_1 and C_2, and k is the mismatch parameter. In addition to the Hamiltonian, Eqs. (1) and (2) conserve the norm (Manley-Rowe invariant), $I = \sum_{m,n}\left(|\psi_{m,n}|^2 + 2\,|\phi_{m,n}|^2\right)$.

Stationary solutions are looked for as $\{\psi_{m,n}(t), \phi_{m,n}(t)\} = \{e^{-i\omega t}\Psi_{m,n}, e^{-2i\omega t}\Phi_{m,n}\}$, where the localized lattice distributions $\Psi_{m,n}, \Phi_{m,n}$ are real for fundamental solitons, and may be complex for more elaborate patterns, such as vortices [17]. To set discrete solitons in motion, one must overcome the PN barrier, i.e., the energy difference between static solitons which are centered, respectively, on a lattice site and between sites. To derive an analytical approximation for the barrier, we consider the continuum limit, with stationary functions Ψ and Φ depending on the radial variable, which is the continuum limit of $r \equiv \sqrt{(m^2+n^2)/C_1}$, and obeying the equations

$$\begin{aligned} \omega\Psi + (\Psi'' + \Psi'/r) + \Psi\Phi &= 0\,, \\ (4\omega + k)\,\Phi + C\,(\Phi'' + \Phi'/r) + \Psi^2 &= 0, \end{aligned} \tag{3}$$

where $C \equiv C_2/C_1$. In this limit, the soliton is approximated by the following ansatz, with $\omega < 0$ and $\chi \equiv (4\omega + k)/C < 0$,

$$\left\{\begin{array}{c}\Psi\\ \Phi\end{array}\right\} = \left\{\begin{array}{c}A\\ B/\sqrt{2}\end{array}\right\}\sqrt{\frac{\sinh\left(2\sqrt{\{|\omega|,|\chi|\}}r\right)}{\sqrt{\{|\omega|,|\chi|\}}r}} \times \operatorname{sech}\left(2\sqrt{\{|\omega|,|\chi|\}}r\right).$$

These expressions have the correct asymptotic form at $r\to\infty$, $\{\Psi,\Phi\} \sim r^{-1/2}\exp\left(-\sqrt{|\{\omega,\chi\}}r\right)$. Substituting the ansatz in Eqs. (3) and demanding its validity at $r \to 0$, one obtains $B = (23/3)|\omega|$, $A = (23/3)\sqrt{\omega\,(4\omega + k)}$.

The Hamiltonian corresponding to axisymmetric solutions of the continuum equations is

$$\begin{aligned} H &= \pi\int_0^\infty rdr\left[2\,(\Psi_r')^2 + C\,(\Phi_r')^2 - 2\Phi\Psi^2 - k\Phi^2\right] \\ &= \pi\int_0^\infty rdr\left(2\omega\Psi^2 + 4\omega\Phi^2 + \Phi\Psi^2\right), \end{aligned} \tag{4}$$

where the derivatives were eliminated using integration by parts and Eqs. (3). To find the PN potential, we apply the lattice discretization to final expression (4) by defining $H_{\rm latt} = \frac{1}{2}\iint\left(2\omega\Psi^2 + 4\omega\Phi^2 + \Phi\Psi^2\right)g(x,y)\,dxdy$, with the grid function,

$$g(x,y) \equiv \sum_{m,n=-\infty}^{+\infty}\delta(x-m)\delta(y-n) = \sum_{p,q=-\infty}^{+\infty}e^{2\pi i(px+qy)}. \tag{5}$$

In the quasi-continuum approximation (which implies small $|\omega|$ and $|\chi|$), the leading terms in $H_{\rm latt}$ correspond to $(p,q) = (\pm 1, 0)$ and $(0, \pm 1)$ in Eq. (5), and yield the PN potential with an exponentially small amplitude,

$$U = (U_0/4)\left[\cos(2\pi\xi) + \cos(2\pi\eta)\right], \tag{6}$$

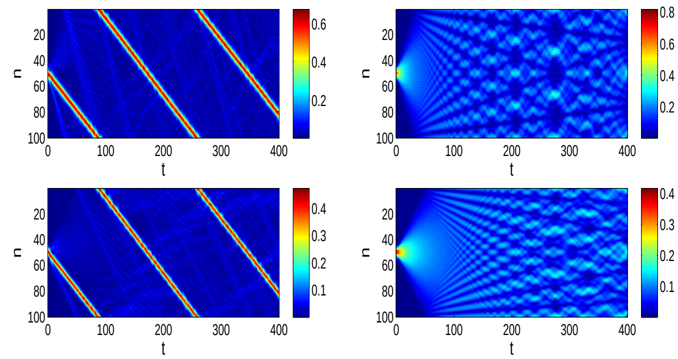

FIG. 1: (Color online) Space-time contour plots of $|\psi_{m,n}|^2$ and $|\phi_{m,n}|^2$ for the FF and SH fields (top and bottom panels) in the 1D lattice with periodic boundary conditions, for $C_1 = C_2 = 1$, $\omega = -0.25$, $k = 0.25$, and shove strengths $S = 0.4$ and 3.0 (left and right panels). The boosted soliton sets in stable motion in the former case, and is destroyed in the latter case.

where (ξ, η) are the coordinates of the soliton's center. In particular, the second term in $H_{\rm latt}$ dominates for $|\chi| > 2|\omega|$, which corresponds to numerical results presented below (with $\omega = -0.25$, $\chi = -0.75$). Then, fitting the slowly varying part of the integrand in $H_{\rm latt}$ to a Gaussian, we obtain the height of PN barrier as $U_0 = -\alpha\left(|\omega|^3/|\chi|\right)\exp\left(-3\pi^2/(10|\chi|)\right)$, with $\alpha \equiv (8\pi/15)23^2 \approx 886$.

Numerical Results. In the 1D and 2D cases alike, we used lattices with periodic boundary conditions, in order to allow indefinitely long progressive motion of solitons. First, we found standing lattice-soliton solutions $\left\{\Psi_{m,n}^{(0)}, \Phi_{m,n}^{(0)}\right\}$, by means of fixed-point iterations. Next, dynamical simulations were initialized by applying a *shove* (kick) to those solutions, which corresponds to initial conditions

$$\{\psi_{m,n}, \phi_{m,n}\} = e^{i\frac{S}{C_{1,2}}(m\cos\theta + n\sin\theta)}\left\{\Psi_{m,n}^{(0)}, \Phi_{m,n}^{(0)}\right\}, \tag{7}$$

where S and θ determine the size and orientation of the shove vector.

Examples of stable motion and destruction of the 1D lattice soliton, to which, respectively, a moderate and strong shove were applied are displayed in Fig. 1, and systematic results, obtained with variation of S and $C_1 = C_2$, are summarized in Fig. 2. Soliton destruction was registered if the kicked soliton would eventually lose $> 30\%$ of its initial norm. For coupling strengths corresponding to Fig. 1, the initial kicks of different sizes S give rise, practically, to two outcomes only, viz., stable motion or destruction. However, for weaker couplings (i.e., stronger discreteness), "localization" is also possible: if S is below a lower critical value, $S_{\rm cr}^{(0)}$, the soliton survives without acquiring any velocity. This outcome is explained by noting that the kinetic energy, $E_{\rm kin} \sim S^2$, imparted to the soliton by the shove, may be insufficient

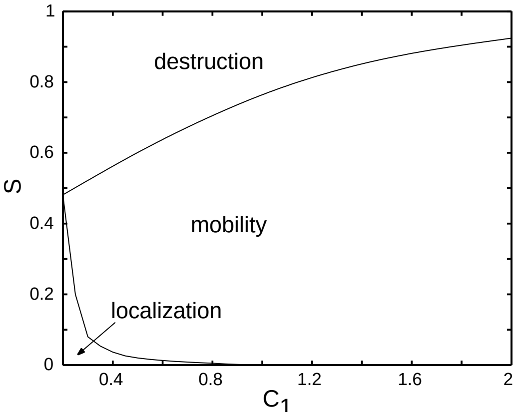

FIG. 2: A diagram in the plane of the coupling strength, $C_1 = C_2$, and shove strength, S, showing different outcomes of kicking the quiescent soliton in the 1D lattice, for $\omega = -0.25$ and $k = 0.25$. ("localization" means that the soliton remains quiescent).

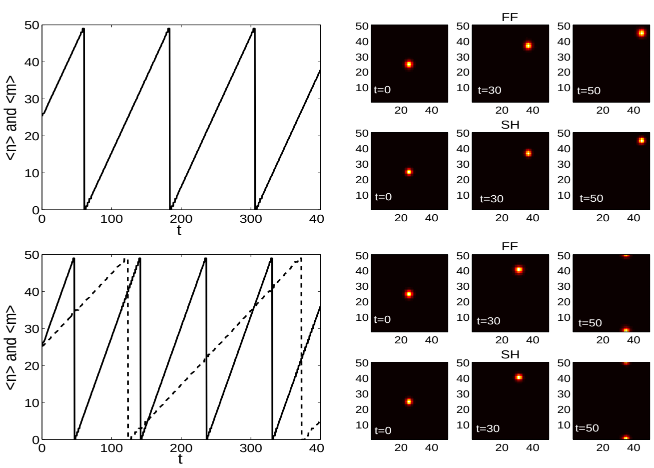

FIG. 3: (Color online) Same as Fig. 1 and with the same parameters, but in the 2D periodic lattice, for the propagation in the diagonal (45 degrees, top panels) and off-diagonal ($\pi/9$ relative to the lattice bonds, bottom panels) directions. The right panels display snapshots of the moving solitons in the FF (top) and SH (bottom) fields at $t = 0, 30, 50$, while the left panels show trajectories of the soliton's center. Dashed and solid line represent the soliton center in the n and m directions. In the diagonal propagation, the two lines coincide.

to overcome the PN barrier; since its height decays exponentially with the increase of the intersite coupling, the "localization" region in Fig. 2 is very small. Thus, general features of the 1D situation are summarized as follows: (i) for $S < S_{\rm cr}^{(0)}$, the soliton remains quiescent; (ii) for $S_{\rm cr}^{(0)} < S < S_{\rm cr}$, it sets in a state of persistent motion; (iii) for $S > S_{\rm cr}$, the soliton is destroyed.

We now turn to the 2D setting, which is more interesting for two reasons. First, as noted above, in the 2D case the mobility is a highly nontrivial feature, absent in the case of the $\chi^{(3)}$ nonlinearity; second, it is interesting to study *anisotropy* of the mobility, i.e., its dependence on the orientation of the initial kick relative to the lattice. Figure 3 shows two examples of stable motion: one along the lattice diagonal, and, to our knowledge, the first ever example of motion in an arbitrary direction (neither diagonal, nor along the bonds) on the lattice.

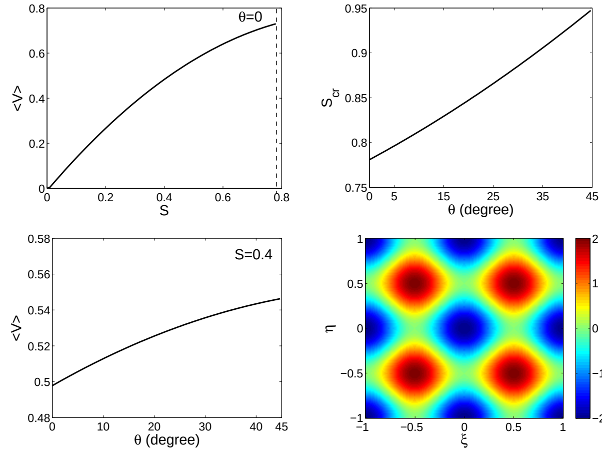

FIG. 4: (Color online) Features of the soliton motion in the 2D periodic lattice, for $C_1 = C_2 = 1$, $k = 0.25$, $\omega = -0.25$. The top left panel shows the velocity versus the shove strength S directed along the lattice bonds (at angle $\theta = 0$); the vertical dashed line indicates the value of $S_{\rm cr}$, beyond which the soliton is destroyed. The top right panel depicts $S_{\rm cr}$ as a function of the orientation of the initial kick, θ. For given $S = 0.4$, the ensuing velocity of the motion is shown versus θ in the bottom left corner. In addition, the bottom right panel shows the analytically predicted PN potential.

In Fig. 4, we summarize the dependence of the mobility on the shove strength, S, and direction, θ, of the initial kick. The "localization" of the 2D soliton (no motion at all) is observed in interval $S < S_{\rm cr}^{(0)}$ ($S_{\rm cr}^{(0)} \approx 0.02$ in the top left panel of Fig. 4). Other generic outcomes again amount to motion at a finite velocity, which depends on S, and destruction by a strong kick, if $S > S_{\rm cr}$.

Particularly noteworthy features, specific to the 2D setting, are presented in the top right and bottom left panels of Fig. 4, viz., dependences of $S_{\rm cr}$ and established velocity on θ. These features demonstrate that the propagation direction easiest to sustain the motion on the square lattice is along the diagonal, as the motion in this direction persists up to the largest value of $S_{\rm cr}$, and is fastest for given S. Both observations may be explained by the fact that the height of the analytically predicted PN potential (6) is smallest along in the diagonal direction, as shown by bottom right panel in Fig. 4. Of course, a lattice solitary wave cannot move straight along the diagonal; however, it may periodically split along the two lattice directions and recombine at the site located diagonally across from the splitting point, which is indeed observed in our numerical data.

We have also examined the situation with $C_2 < C_1$, and obtained similar results, but with larger $S_{\rm cr}^{(0)}$. In the special case of $C_2 = 0$ (no coupling in the SH field), moving solitons cannot be generated, which is easy to explain: with $C_2 = 0$, Eq. (2) yields $\Phi_{m,n} = -\Psi_{m,n}^2/(4\omega+k)$, and the substitution of this in Eq. (1) makes the model equivalent to one with the cubic nonlinearity, where steadily moving 2D discrete solitons do not exist.

Conclusions. In this work, we examined the mobility of solitons in 1D and 2D lattices with the quadratic nonlinearity. We have shown that the solitons feature stable motion much easier than their counterparts in 1D lattices with the cubic nonlinearity, and they may also be mobile on the 2D lattice, where the cubic solitons cannot move at all. In the 2D lattice, we have for the first time reported a possibility of motion of the soliton in an arbitrary direction (neither axial nor diagonal), the motion along the diagonal being most persistent. A qualitative explanation for these features was provided by the analytical approximation for the 2D Peierls-Nabarro potential.

It may be interesting to extend the analysis to other 1D and, especially, 2D models, where mobile solitons may be expected, such as systems with competing nonlinearities (the cubic-quintic model [34], or the Salerno model with competing on-site and inter-site cubic terms [35]). A full proof of the existence of traveling lattice solitons is a challenging computational [36] and mathematical [37] problem.

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