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Mode-locking of a driven Bose-Einstein condensate

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We consider the dynamics of a driven Bose-Einstein condensate with positive scattering length. Employing an accustomed variational treatment we show that when the scattering length is time-modulated as $a\left(1+\epsilon\sin\left(\omega\left(t\right)t\right)\right)$, where $\omega\left(t\right)$ increases linearly in time, *i.e.*, $\omega\left(t\right)=\gamma t$, the response frequency of the condensate locks to the eigenfrequency for small values of ϵ. A simple analytical model is presented which explains this phenomenon by mapping it to an auto-resonance, *i.e.*, close to resonance the reduced equations describing the collective behavior of the condensate are equivalent to those of a particle trapped in a finite-depth energy-minimum of an effective potential.

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I. INTRODUCTION

Ever since the experimental attainment of Bose-Einstein condensates (BECs) back in 1995 [1] there has been a surge of research on this topic. The reason for this scientific effervescence is to be found in the almost unprecedented experimental maneuverability of these ultracold gases which engrossed scientists from many distinct fields such as nonlinear dynamics, quantum and nonlinear optics, nuclear and condensed matter physics, to name just a few. Equally appealing are the theoretical insights into the dynamics of Bose-condensed gases obtained through the so-called Gross-Pitaevskii equation (GPE) [2], a cubic Schrödinger equation where the nonlinearity accounts at mean-field level for the inter-atomic interactions close to absolute zero temperature.

Among the most notable results on the nonlinear side of BECs are the theoretical prediction and the subsequent experimental realization of distinct soliton classes and detailed accounts of their underlying dynamics, see [3–7] and references therein for some of the recent developments, and the nonlinear infringement of Bloch's periodicity condition. The latter implies that a condensate loaded into the so-called optical lattice, *i.e.*, a periodic potential generated by two counter-propagating laser beams, can have a periodic spatial profile with a period different than that of the underlying lattice [8]. Other works include investigations into the parametric resonances exhibited by a Bose-condensed gas whose scattering length is time-modulated on a constant frequency close to the eigenfrequency of the system (see [9–11], and references therein).

It is the purpose of this paper to investigate the nonlinear dynamics of a repulsive Bose-condensed gas whose scattering length is time-modulated on a linearly increasing frequency, *i.e.*, a scattering length of the type $a\left(1+\epsilon\sin\left(\omega\left(t\right)t\right)\right)$, where $\omega\left(t\right)=\gamma t$. While cubic Schrödinger equations have been scrutinized in nonlinear optics over the past few decades, it is, however, only in BECs that one can take advantage of the so-called Feshbach resonances [12] to control the frequency on which the cubic term (*i.e.*, the scattering length) is modulated.

Employing a habitual variational treatment [14] we reduce the dynamics of a three-dimensional fully-symmetric condensate to only one ordinary differential equation (ODE). Our main finding is that, for small values of ϵ, the condensate mode-locks to the eigenfrequency, a peculiarity that disappears for high values of ϵ. As shown in Section IV, the condensate is collectively described by a single reduced ODE on the condensates' width that emulates the scenario of a particle trapped in the finite-depth energy-minimum of an effective potential well.

The paper is structured as follows: Section II is dedicated to the GPE and to the variational method that simplifies the condensate dynamics to an ODE. Section III gives numerical results on frequency locking, while Section IV puts forward a simple analytical model that explains the locking through the so-called auto-resonance. Section V gathers our conclusions.

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II. THE GROSS-PITAEVSKII EQUATION

The dynamics of Bose-condensed gases close to absolute zero temperature is accurately described by the Gross-Pitaevskii equation. In three spatial dimensions it reads

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(r)\psi + U\left|\psi\right|^2\psi, \tag{1}$$

where $V(r)$ is the trapping potential, here taken as

$$V(r) = \frac{m}{2}\,\lambda^2 r^2, \tag{2}$$

where $f = \lambda/2\pi$ is the trap frequency (in Hz) that fixes the strength of the magnetic trap. For simplicity we restrict our analysis to symmetric potentials. The coefficient of the cubic term is $U = 4\pi\hbar^2 a/m$, where a is the two-body scattering length and m the atomic mass.

Due to the intricate nature of the GPE we shall further simplify the problem by restricting ψ to an amenable family of trial functions and study the time evolution of the parameters that define it. This reduces the infinite-dimensional problem of solving Eq. (1) to solving an ODE. A natural choice for the trial function (see [13, 14] for the BEC results, and [15, 16] for similar calculations carried out in nonlinear optics), which actually corresponds to the exact solution in the linear limit ($U = 0$), is the three-dimensional Gaussian-like profile:

$$\psi(r,t) = A(t)\exp\left[-\frac{r^2}{2w^2} + ir^2\beta\right], \tag{3}$$

where A is the wave-function amplitude at the center of the cloud, w is the width of the condensate while β, the so-called chirp, is the canonical conjugate of w.

After the classical variational recipe (see [14] for details) the ansatz gives rise to the following equation in the width of the condensate

$$\frac{d^2}{dt^2}w + \lambda^2 w = \frac{\hbar^2}{m^2}\frac{1}{w^3} + \frac{U}{2\sqrt{2}m}\frac{N}{\pi^{3/2}w^4}. \tag{4}$$

Notice that this equation holds for both time-dependent and time-independent scattering length, *i.e.*, the equation is left unchanged when $U \to U(t)$. It is worth mentioning that the equation for the width does not depend on the chirp β and that, in turn, the equation for β is driven by the width.

Introducing $P = \sqrt{2/\pi}Na/a_0$, $\tau = \lambda t$ and the rescaled width $v = w/a_0$, where $a_0 = \sqrt{\hbar/m\lambda}$, Eq. (4) reads

$$\frac{d^2}{d\tau^2}v + v = \frac{1}{v^3} + \frac{P}{v^4}. \tag{5}$$

Around the equilibrium point $\tilde{v}$, defined implicitly by

$$\tilde{v} = \frac{1}{\tilde{v}^3} + \frac{P}{\tilde{v}^4}, \tag{6}$$

the dynamics of the width of the condensate $v = \tilde{v} + \delta$ is given by

$$\frac{d^2}{d\tau^2}\delta + \delta\left(1 + \frac{3}{\tilde{v}^4} + \frac{4P}{\tilde{v}^5}\right) = 0, \tag{7}$$

indicating a period of $T = 2\pi/\omega_P$ where the natural eigenfrequency of the system is

$$\omega_P = \left(1 + \frac{3}{\tilde{v}^4} + \frac{4P}{\tilde{v}^5}\right)^{1/2}. \tag{8}$$

III. MODE LOCKING

Sweeping linearly the frequency of our driving field, which in turn gives a scattering length that goes like $a\left(1 + \epsilon\sin\left(\gamma\tau^2\right)\right)$, we are faced with the nonlinear ODE

$$\frac{d^2}{d\tau^2}v + v = \frac{1}{v^3} + \frac{P}{v^4}\left(1 + \epsilon\sin\left(\gamma\tau^2\right)\right). \tag{9}$$

Equation (9) is solved through an embedded Runge-Kutta method that uses a 4-5 Dorman-Prince pair [17].

Our main finding is that for small values of ϵ the width of the condensate shows periodic oscillations of constant amplitude (physically a breathing mode) whose frequency is equal to the eigenfrequency of the system ω_P (cf. Eq. (8)). We call this process *mode-locking* [18]. As demonstrated in the next section, mode-locking amounts to the condensate being placed in an energy minimum of the system, a situation which is lost for strong driving fields due to the finite depth of the energy-minimum well.

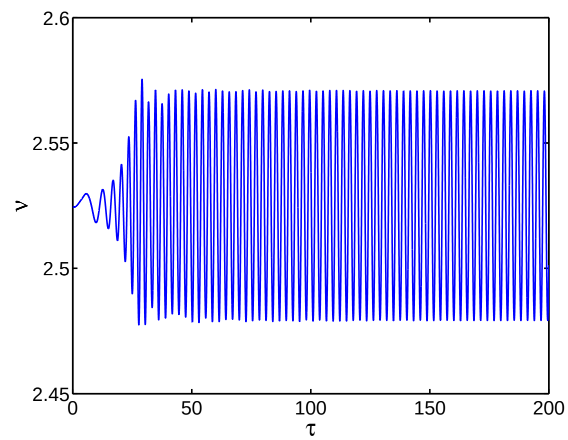

Figure 1: The dynamics of the width of the condensate for $P = 100$, $\epsilon = 0.01$ and $\gamma = 0.05$. The condensate starts to respond periodically at $\tau_c = \omega_P/2\gamma \approx 22$; the time needed to stabilize the amplitude is, however, slightly longer. This example corresponds to $24,500$ ^{23}Na loaded in a magnetic trap with frequency 159Hz and τ is measured in milliseconds.

Locking phenomena go a long way back: as early as the 17th century the Dutch physicist Christian Huygens noted that two clocks hanging back-to-back on the wall tend to synchronize their motion. This type of locking is generally present in dissipative systems with competing frequencies. The two frequencies may arise dynamically within the system (as for the two clocks) or through the coupling of an oscillating motion to an external periodic force. In the case of a magnetically trapped BEC (which is usually regarded/modelled as a non-dissipative system) we find that the frequency sweep entailed by the $\sin(\gamma t^2)$ term gives rise to a breathing mode whose frequency is equal to the natural frequency of the system.

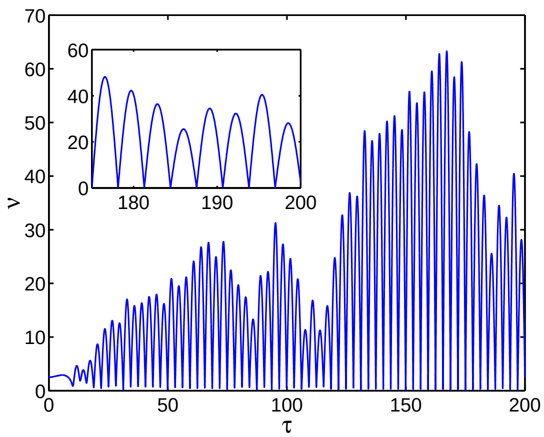

Figure 2: The dynamics of the width of the condensate for $P = 100$, $\epsilon = 1.0$ and $\gamma = 0.05$. Notice the chaotic response of the condensate and the bouncing ball behavior shown in the inset. Same rescaling as in Fig. 1 so that τ is measured in milliseconds.

We illustrate our results on mode-locking by showing the dynamics of the width of the condensate for $P = 100$. We purposedly chose parameter values that would yield experimental feasible situations. For example, if one considers a magnetic trap with frequency $f \approx 159$Hz (*i.e.*, $\lambda = 2\pi f = 1000$) for a dilute BEC [13] of approximately 24,500 ^{23}Na, the adimensionalization yields a temporal rescaling $\tau = 1{,}000\, t$ such that, in all our examples, time τ is measured in milliseconds. In Fig. 1 we have plotted the dynamics of the width for $\epsilon = 0.01$ and $\gamma = 0.05$. This is a typical mode-locking dynamics that includes two distinct regions (see Fig. 1): *i.)* a transient regime during which the response frequency of the condensate locks to the eigenfrequency of the system, and *ii.)* the mode-locked part when the width of the condensate shows periodic oscillations of constant amplitude. The critical time τ_c needed to see response on the system's eigenfrequency may be obtained by writing $t = \tau_c + \Delta t$ ($|\Delta t| \ll 1$) and expanding the argument of the drive γt^2 yields an effective frequency $\omega_P = 2\gamma\tau_c$, and thus $\tau_c = \omega_P/2\gamma$. Namely, the transient fades out when the effective frequency of the driving field matches the eigenfrequency of the system. Notice, however, that the time needed to stabilize the amplitude is slightly longer.

In Fig. 2 we have plotted the dynamics of the condensate's width for a relatively large value of $\epsilon = 1.0$ and $\gamma = 0.05$. This is the typical dynamics of the condensate when frequency locking is lost due to the high strength of the driving field. Notice the "bouncing ball" behavior, a peculiarity of the dynamics of the system for high ϵ due to the $1/v^3$ singularity [14].

In Fig. 3 we have plotted the dynamics of the condensate's width for an intermediate value of $\epsilon = 0.065$ and $\gamma = 0.01$ to illustrate the dynamics outside the linear regime [19]. The nonlinear regime shows two distinct features: *i.)* while the observed frequency shows only slight deviations from Eq. (8), the shape of the oscillations shows contributions from the higher harmonics, *i.e.*, strong deviation from the shape of a sine-wave, and *ii.)* in addition to the mode-locking seen approximately at $\tau_c \approx \omega_P/2\gamma$ there is a series of additional super-harmonic nonlinear resonances approximately at $\tau_c = n\omega_P/2\gamma$, where n is an integer larger than one (see [20]). This latter feature falls outside the simple model put forward in the next section which only accounts for the linear process, *i.e.*, the mode-locking at $\tau = \omega_P/2\gamma$.

Finally, in Fig. 4 we have plotted the dynamics of the condensate's width for $\epsilon = 0.065$ and small value of $\gamma = 0.001$. For this small value of γ the resonances are seen to appear exactly at the predicted $\tau_c = n\omega_P/2\gamma \approx 1115n$ (even for large values of n).

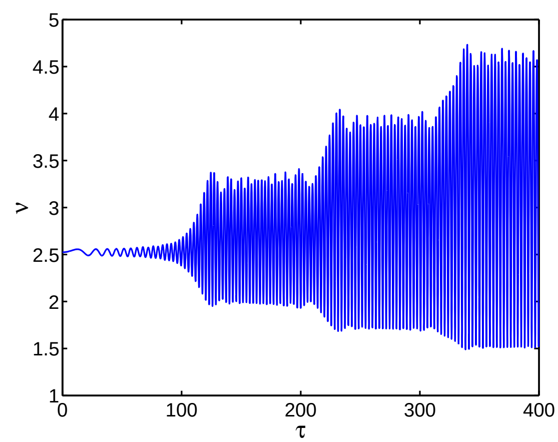

Figure 3: The dynamics of the width of the condensate for $P = 100$, $\epsilon = 0.065$ and $\gamma = 0.01$. The condensate starts to respond periodically approximately at $\tau_c = \omega_P/2\gamma \approx 111$ but there are additional super-harmonic nonlinear resonances at approximately $\tau_c = n\omega_P/2\gamma$, where $n = 2, 3, 4\ldots$. Same rescaling as in Fig. 1 so that τ is measured in milliseconds.

While the reported numerics rely on variational computations we have found the same qualitative behavior in the GPE [21]. Finally, it is important to notice that the frequency locking reported in this paper takes places both for the ground state of the condensate and its excited states. Solitons for instance are excited into breather states by the driving field as will be shown elsewhere [21].

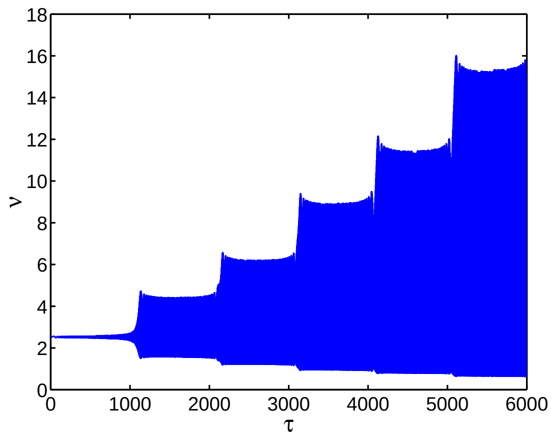

Figure 4: The dynamics of the width of the condensate for $P = 100$, $\epsilon = 0.065$ and $\gamma = 0.001$. For such a small γ value, the condensate responds periodically precisely at $\tau_c = n\omega_P/2\gamma \approx 1115n$. Same rescaling as in Fig. 1 so that τ is measured in milliseconds.

IV. ANALYTIC MODEL

Following the recipe of auto-resonance phenomena put forward by Friedland and collaborators (see [22], and references therein) we recast v as

$$v = \tilde{v} + \delta(\tau), \tag{10}$$

where $\tilde{v}$ is the equilibrium width (cf. (6)) and $\delta(\tau)$ measures the deviation from equilibrium. Refraining to the linear regime, *i.e.*, $\delta(\tau) \ll 1$, Eq. (9) reduces to

$$\frac{d^2\delta}{d\tau^2} + \omega_P^2\delta = \frac{\epsilon P}{\tilde{v}^4}\sin\left(\gamma\tau^2\right),$$

where we have discarded a term linear in $\epsilon\delta$ for being second order.

We study the dynamics close to resonance and show that the condensate collectively behaves like a particle loaded in a tilted cosine-like effective potential. We show that mode-locking is equivalent to the effective dynamics of the particle trapped in an energy-minimum of the potential.

Taking $\delta = a(\tau)\sin\varphi(\tau)$ and discarding the second derivative of a with respect to τ, the so-called adiabatic assumption, one has

$$i2\dot{a}\dot{\varphi} + ia\ddot{\varphi} - a\dot{\varphi}^2 + \omega_P^2 a = \frac{\epsilon P}{\tilde{v}^4}\exp\left(i\gamma\tau^2 - i\varphi\right),$$

where $\dot{a} = da/d\tau$. Equating real and imaginary parts we obtain

$$a\omega_P^2 - a\dot{\varphi}^2 = \frac{\epsilon P}{\tilde{v}^4}\cos(\gamma\tau^2 - \varphi) \tag{11}$$

for the real part, while the imaginary one gives

$$2\dot{a}\dot{\varphi} + a\ddot{\varphi} = \frac{\epsilon P}{\tilde{v}^4}\sin(\gamma\tau^2 - \varphi). \tag{12}$$

Refraining now to the case close to the resonance, *i.e.*, we limit the analysis to a vicinity of τ_c such that

$$\dot{\varphi}(\tau_c) \simeq \omega_P, \quad \text{and} \quad \ddot{\varphi}(\tau_c) \simeq 0,$$

the previous equations yield

$$\omega_P - \dot{\varphi} = \frac{\epsilon P}{2\omega_P a\tilde{v}^4}\cos(\gamma\tau^2 - \varphi) \tag{13}$$

and

$$\frac{d}{d\tau}\left(a^2\right) = \frac{a\epsilon P}{\omega_P\tilde{v}^4}\sin(\gamma\tau^2 - \varphi). \tag{14}$$

Defining the action $I = a^2$ and the phase mismatch $\Phi = \gamma\tau^2 - \varphi$ variables we can recast the previous equations as

$$\dot{\Phi} = 2\gamma\tau - \omega_P + \frac{\epsilon P}{2\omega_P\sqrt{I}\tilde{v}^4}\cos\Phi \tag{15}$$

and

$$\dot{I} = \frac{\sqrt{I}\epsilon P}{\omega_P\tilde{v}^4}\sin\Phi. \tag{16}$$

In order for the condensate to stay mode-locked Φ must be close to 0 or π and the right hand side of Eq. (15) should be equal to zero, *i.e.*,

$$\dot{\Phi}(\tau_c) = 2\gamma\tau_c - \omega_P + \frac{\epsilon P}{2\omega_P\sqrt{I_0}\tilde{v}^4}\cos\tilde{\Phi} = 0,$$

where I_0 is the equilibrium action while $\tilde{\Phi}$ is the equilibrium phase-mismatch. Notice that $\dot{\Phi} = 0$ amounts to $\tau_c = \omega_P/2\gamma$. The solution of interest is $\tilde{\Phi} = \pi$, for $\tilde{\Phi} = 0$ corresponds to an energy maximum (see below). Setting $I = I_0 + \Delta$ and $\Phi = \tilde{\Phi} + \phi$, where Δ and ϕ are small, the dynamics around the equilibrium is given by the following Hamiltonian system

$$\begin{cases} \dot{\phi} &= \Delta S \\ \dot{\Delta} &= -A\sin\phi + \dfrac{2\gamma}{S}, \end{cases}$$

where $S = \epsilon P/4\omega_P\tilde{v}^4 I_0^{3/2}$ and $A = \sqrt{I_0}\epsilon P/\omega_P\tilde{v}^4$. The associated Hamilton's function is

$$\mathcal{H}(\Delta, \phi) = \frac{S\Delta^2}{2} + V_1(\phi),$$

where the potential is given by

$$V_1(\phi) = -A\cos\phi - \frac{2\gamma\phi}{S}.$$

If one is to linearize around $\tilde{\Phi} = 0$ the ensuing potential would be

$$V_2(\phi) = A\cos\phi + \frac{2\gamma\phi}{S},$$

which has a maximum for $\phi = 0$ and not a minimum as V_1 has.

The frequency locking reported in the previous section is now transparent: around $\Phi = \pi$ there is an energy minimum corresponding to the system oscillating on its eigenfrequency, while around $\Phi = 0$ there is an energy maximum of no physical interest. In the light of our linear model we infer that due to the finite depth of the energy-minimum well at $\Phi = \pi$, there is a critical strength of the driving field above which the frequency locking is lost. This process is illustrated in Fig. 2.

V. CONCLUSIONS

In this paper we have shown by means of a variational treatment that modulating the scattering length of a trapped BEC with repulsive interactions as $a\left(1 + \epsilon\sin\left(\gamma t^2\right)\right)$, leads to the locking of its response frequency to the eigenfrequency for small values of ϵ. Physically, the mode-locking amounts to a breathing mode whose frequency is equal to the natural frequency of the system. To the best of our knowledge this is the first paper to analyze mode-locking in a BEC context. In order to exhibit the physical mechanism behind it we have restricted ourselves to a variational calculation that captures the main dynamics. To this end we have used a simple analytical model and showed that the equations describing the collective behavior of the condensate are equivalent to those of a particle trapped in a finite-depth energy-minimum of a potential.

Future research should be focused on asymmetric three-dimensional condensates and to analyzing the interplay between the inherent mode-locking processes that take place. Also on the side of future research lies the dynamics of the condensate for negative scattering lengths and the extension of the current observations to multi-component condensates.

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