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# Existence of Steady State Bright Vortex Solutions to the Cubic-Quintic Nonlinear Schrödinger Equation

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**Abstract:** We seek vortex solutions to the Cubic-Quintic Nonlinear Schrödinger Equation both analytically and numerically. Our analytical approach is based on the onedimensional exact solution of a steady-state profile solution, combined with a variational approach. Our numerical solutions are formed by using our analytical results as an initial guess for a nonlinear modified Gauss-Newton optimization routine. Our results show that the existence region for the vortices, which have been previously deputed, are in fact the same as the known existence regions for the one-dimensional profile.

**Keywords:** vortices, nonlinear Schrödinger equation, cubic-quintic nonlinearity.

#### 1 Introduction

The Cubic-Quintic Nonlinear Schrödinger Equation (CQNLS) can be used to model a variety of physical systems. These include light propagation through nonlinear optical media such as double-doped optical fibers [3] and non-Kerr crystals [4]. It has also been used in various contexts of Bose-Einstein condensates. The key feature of CQNLS models is the competition of the focusing and defocusing nonlinear terms. This allows for stable structures which would otherwise be unstable in the cubic NLS.

An interesting property of light is that of twisted light. This is when light is twisted about its axis, and has a topological charge which tells how many times it twists around for each wavelength. Such light is called an optical vortex, and has the property that the center is identically zero in intensity, and hence looks like a ring of light. A higher charge vortex will result in more angular momentum, and thus a wider radius to the ring as shown in Fig. 1.



Figure 1: Projection of optical vortices of different topological charges onto a dark surface formed by passing a laser beam through a specialized fork grating. Photo courtesy of A. Hansen, Stony Brook University.

When such vortices propagate through a nonlinear medium such as certain crystals or fibers with a cubic Kerr-type effect, they exhibit azimuthal modulational instability (MI) in which they break up into filaments, thus excluding their practical use. An example of this is shown in Fig. 2.



Figure 2: Example of a vortex of charge 4 in the CQNLS exhibiting break up due to azimuthal modulational instability. Shown is the intensity depicted by the modulus-squared of the wave-function.

However, in materials which have competing self-focusing and defocusing nonlinear properties, the vortices can be stable. The CQNLS can model such vortices.

Our goal in this report is to find and describe the vortex solutions to the CQNLS, and to determine their existence bounds. This is important because others have shown that in order for vortices to be MI stable, they must have a large value of complex frequency. Knowing the maximum possible frequency of a vortex solution will help in determining if vortices of higher charges can be MI stable. There are some potential applications of finding stable optical vortices in cryptography, quantum computing, and also data compression. Since in principle, the charge of the vortex can be made to take any integer value, a single pulse of a light vortex can transmit one of hundreds of possible messages.

## 2 Review of One-Dimentional Solutions

We first review the steady state profile solution to the one-dimensional CQNLS. The undimentionalized one-dimensional CQNLS takes the form:

$$i\Psi_t + \Psi_{xx} + |\Psi|^2 \Psi - |\Psi|^4 \Psi = 0,$$
 (1)

while the one-dimensional steady state wave profile takes the form:

$$\Psi(x,t) = f(x) e^{i\Omega t}.$$
 (2)

When Eq. (2) is inserted into Eq. (1), we get:

$$-\Omega f + \frac{d^2 f}{dx^2} + f^3 - f^5 = 0.$$
 (3)

Eq. (3) can be solved for f(x), which yields a soliton-like profile [1]:

$$f^2 = \frac{4\Omega}{1 + \sqrt{1 - (16/3)\Omega} \cosh(2\sqrt{\Omega}x)}.$$
 (4)

It is apparent that the solution does not exist for all values of  $\Omega$ . The maximum value that  $\Omega$  can take is:

$$\Omega_{\rm max}^{\rm 1D} = 3/16 = 0.1875. \tag{5}$$

Therefore, there is an existence region of  $\Omega \in [0, \Omega_{\max}^{1D}]$ , where when  $\Omega = \Omega_{\max}^{1D}$ , the solution is simply a constant value of  $f(x) = f_0 = \sqrt{3/4}$ , and  $\Omega = 0$  yields the trivial solution. This solution describes a sech-shaped curve which flattens out as  $\Omega \to \Omega_{\max}^{1D}$ . The family of solutions are depicted in Fig.3.



Figure 3: Depictions of steady state solitary solutions for the one-dimensional CQNLS.

## 3 Two-Dimensional CQNLS Vortex Profile

The non-dimensionalized two-dimensional CQNLS is written as:

$$i\Psi_t + \nabla^2 \Psi + s_1 |\Psi|^2 \Psi + s_2 |\Psi|^4 \Psi = 0, \quad (6)$$

where  $\nabla^2 \Psi$  is the two-dimensional Laplacian of the wave function and  $s_i = \pm 1$ . Since the natural coordinate system for studying vortices is polar, the Laplacian takes on the well-known form:

$$\nabla^2 \Psi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2}.$$
 (7)

For bright vortices (which we will exclusively analyze),  $s_1 = +1$  and  $s_2 = -1$ .

A steady state vortex in the CQNLS can be described in general as:

$$\Psi(r,\theta,t) = f(r) e^{i(m\theta + \Omega t)}, \qquad (8)$$

where  $f(r) \in \Re$  is the steady state radial profile, *m* is the topological charge, and  $\Omega$  is the complex frequency. An example of such a vortex is shown in Fig. 4



Figure 4: Depiction of a vortex of charge 5 and complex frequency of 0.16 in the CQNLS. The 'height' of the vortex represents the value of the wave-function. The gray area represents the intenisty, while the blue and red meshes are the real and imaginary parts of the wave-function respectivly.

Inserting Eq. (8) Eq. (6) yields:

$$-\left(\Omega + \frac{m^2}{r^2}\right)f + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + f^3 - f^5 = 0.$$
(9)

If we assume that the vortex has a large radius, then the region of interest in the evolution equation has r >> 1, in which case the variations in the variable r behave like a constant denoted  $r_c$ , which we take to be the center of the radial profile. With this assumption, Eq. (9) becomes:

$$-\left(\Omega + \frac{m^2}{r_c^2}\right)f + \frac{\partial^2 f}{\partial r^2} + f^3 - f^5 = 0, \ (10)$$

which is exactly the same form as the onedimensional CQNLS profile ODE depicted in Eq. (3), but with an offset  $\Omega$  value. We designate this offset  $\Omega$  as:

$$\Omega^* = \Omega + \frac{m^2}{r_c^2},\tag{11}$$

and can then rewrite Eq. (10) as:

$$-\Omega^* f + \frac{\partial^2 f}{\partial r^2} + f^3 - f^5 = 0.$$
 (12)

The steady state profile for Eq. (12), is simply the same as in the one-dimensional case:

$$f_{\rm asy}^2 = \frac{4\Omega^*}{1 + S \,\cosh\left(2\sqrt{\Omega^*} \left(r - r_c\right)\right)},\quad(13)$$

where  $S = \sqrt{1 - (16/3)\Omega^*}$ .

What we need now is an expression for the central radius of the profile,  $r_c$  and  $\Omega^*$ . Finding one of the two automatically gets us the other from the expression in Eq. (11).

#### 3.1 Variational Approach

We start by inserting the general vortex solution of Eq. (8) into the Lagrangian density of the CQNLS:

$$\mathcal{L} = \left(\Omega + \frac{m^2}{r^2}\right)f^2 + \left(\frac{df}{dr}\right)^2 - \frac{1}{2}f^4 + \frac{1}{3}f^6.$$
(14)

Since f(r) is steady state, the radial integrals over f(r) in the Lagrangian will become constants (which will depend on m and  $\Omega$ ), and we can write it as:

$$L = 2\pi [\Omega C_1 + m^2 C_3 + C_2 - \frac{1}{2} C_4 + \frac{1}{3} C_5],$$
(15)

where,

$$C_1 = \int_0^\infty f(r)^2 r \, dr, \qquad (16)$$

$$C_2 = \int_0^\infty \left(\frac{df}{dr}\right)^2 r \, dr, \qquad (17)$$

$$C_3 = \int_0^\infty \frac{1}{r^2} f(r)^2 r \, dr, \qquad (18)$$

$$C_4 = \int_0^\infty f(r)^4 r \, dr, \tag{19}$$

$$C_5 = \int_0^\infty f(r)^6 r \, dr.$$
 (20)

The Euler-Lagrangian equations take the form:

$$\frac{\partial L}{\partial \Omega^*} = 0, \qquad \frac{\partial L}{\partial r_c} = 0, \qquad (21)$$

Evaluating Eq. (21) leads to two equations for  $\Omega^*$  and  $r_c$ . Solving the equations we get:

$$r_{c}^{\text{va}} = m \left[ \Omega - \frac{3}{16} + \frac{1}{2T} \sqrt{\frac{3}{16} \Omega^{*}} \right]^{-1/2}, (22)$$
$$\Omega^{*} = \Omega + \frac{m^{2}}{(-\sqrt{2})^{2}}, (23)$$

$$\Omega^* \qquad \qquad = \Omega + \frac{m^2}{(r_c^{\mathrm{va}})^2}, \qquad (23)$$

where T is defined as:

$$T = \operatorname{arctanh}\left[\sqrt{\frac{3}{16\Omega^*}} - \sqrt{\frac{3}{16\Omega^*} - 1}\right],$$
(24)

Obviously, to solve for  $r_c$  analytically in terms of  $\Omega$  seems impossible. However, it is possible to solve for  $r_c$  numerically using a root finder routine (in our case a simple bisection method with a tolerance of  $10^{-15}$ ), and then using the result to obtain the desired VA profile.

There is a nice analytical result that can be obtained from Eq. (22). If we combine the two equations, we can rearrange things to get an analytical form for  $\Omega$  as a function of  $\Omega^*:$ 

$$\Omega = G(\Omega^*) = \frac{\Omega^*}{2} + \frac{3}{32} - \frac{1}{4T}\sqrt{\frac{3}{16}\Omega^*}, \quad (25)$$

where T is defined as in Eq. (24). We can write  $r_c$  in terms of  $G(\Omega^*)$  as:

$$r_c^{va} = \frac{m}{\sqrt{\Omega^* - G(\Omega^*)}},\tag{26}$$

which is simply a rearrangement of Eq. (11). With this formulation of the VA, we can easily see how  $\Omega$  and  $\Omega^*$  are related, as well as how  $r_c^{\rm va}$  behaves. It should be noted, that it seems that while  $r_c^{\rm va}$  depends on m, the relationship between  $\Omega$  and  $\Omega^*$  does not. This means that we have a relationship between  $\Omega$  and  $\Omega^*$  does not. This means that we have a relationship between  $\Omega$  and  $\Omega^*$  that is valid for all charges m (keeping our large-r approximation in mind). In Fig. 5 we show  $r_c^{\rm va}$  versus  $\Omega$  and  $\Omega^*$  with m = 5.



Figure 5:  $r_c^{\text{va}}$  versus  $\Omega = G(\Omega^*)$  and  $\Omega^*$  for m = 5.

We see that as one increases  $\Omega$  (and  $\Omega^*$ ), the radius of the vortex starts out very large (infinity for  $\Omega = 0$ ), decreases until it reaches a minimum value, and then increases rapidly (it becomes infinite at  $\Omega = \Omega_{\max}^{2D}$ ).

## 3.2 Existence Bound for Two-dimensional Vortex Profiles

We can glean some interesting information from Eq. (25). From numerical investigations done in [7] and [2], the value of the existence criteria in the 2D vortex solution was stated as  $\Omega_{\rm max}^{\rm 2D} \approx 0.180$ . Later it was shown analytically in [6] that:

$$\Omega_{\max}^{2D} = \Omega_{\max}^{1D} = 3/16 = 0.1875.$$
(27)

It is possible to explain why the numerical results were incorrect by looking at the relationship between  $\Omega$  and  $\Omega^*$  given by Eq. (25). In Fig. 6 we plot  $\Omega = G(\Omega^*)$  versus  $\Omega^*$  for  $\Omega^* \in [0.01, 0.1875]$ .



Figure 6:  $\Omega$  vs.  $\Omega^*$  for the VA ansatz.

We see that as  $\Omega^* \to 3/16$ ,  $\Omega$  approaches 0.1875, but appears to only get to around 0.171 before jumping to 0.1875. We know that as  $\Omega^* \to 0.1875$ ,  $\Omega \to 0.1875$  because we can see from Eq. (25), that as  $\Omega^* \to 3/16$ ,  $T \to \infty$ , and so  $\Omega \to \Omega^*/2 + 3/32 = 0.1875$ . The reason why there is a 'jump' in Fig. 6 is due to the extreme sensitivity of the relationship between  $\Omega$  and  $\Omega^*$  near  $\Omega_{\text{max}}^{2D}$ . In Table. 1 we show some example values of  $\Omega^*$ their corresponding  $G(\Omega^*) = \Omega$ .

$\Omega^*$	$\Omega = G(\Omega^*)$
0.1874	0.1664
0.187499	0.1736
0.187499999	0.1783
0.187499999999	0.1806
0.187499999999999999	0.1823

Table 1: Evaluation of  $\Omega = G(\Omega^*)$  near 0.1875. We only show four significant digits on  $G(\Omega^*)$ .

As is clearly seen, one quickly approaches the limit of machine precision in  $\Omega^*$  as  $\Omega \rightarrow 3/16$ . It is not surprising then, that numerical estimates of  $\Omega_{\max}^{2D}$  (done by using a shooting method to try to find profiles at different  $\Omega$  values) were inaccurate. We will confirm the existence bound of  $\Omega_{\max}^{2D} = 3/16$  by computing some vortex profiles with  $\Omega > 0.18$ .

## 4 Numerically-Exact Steady State Vortex Profiles

Here we use describe our method for finding numerically-exact vortex profiles.

#### 4.1 Numerical Nonlinear Optimization

We recall that the goal is to find the radial profile f(r) which satisfies Eq. (9) for a given m and  $\Omega$ . If we discretize the radial direction and use a second order finite difference approximation for the derivatives, we can write Eq. (9) as the vector function:

$$\vec{F}(\vec{f}) = -\left(\Omega + \frac{m^2}{r_j^2}\right)f_j + \frac{1}{r_j\Delta r}\left(r_{j+1/2}\frac{f_{j+1}-f_j}{\Delta r} - r_{j-1/2}\frac{f_j-f_{j-1}}{\Delta r}\right) + f_j^3 - f_j^5 = \vec{0},$$

where  $\Delta r$  is the grid spacing length,  $r_j = j\Delta r$ , and  $f_j = f(r_j)$ .

What we want is a discrete radial profile input vector  $(\vec{f}^*)$  which satisfies  $\vec{F}(\vec{f}^*) = \vec{0}$ . The problem can be looked at as a minimization problem, which we solve using a numerical optimization technique known as a modified Gauss-Newton method [5].

We iterate a trial solution  $(\vec{f}_0)$  of Eq. (28) towards a local minimum solution,  $\vec{f^*}$ , by taking steps defined as:

$$\vec{f}_{k+1} = \vec{f}_k + \alpha_k \vec{p}_k,$$

where  $\alpha_k$  is the step length for step number k, and  $p_k$  is the step direction. To mark our progress towards  $\vec{F}(\vec{f}^*(r)) = \vec{0}$  we define a merit function as:

$$M(\vec{f}) = \frac{1}{2} \sum_{i=1}^{n} (F_i(\vec{f}))^2, \qquad (28)$$

the gradient of which is:

$$\nabla M(\vec{f}) = J(\vec{f})^T \vec{F}(\vec{f}).$$
<sup>(29)</sup>

To determine the step size  $\alpha_k$  we use an inexact line search which takes the step direction and computes (using backtracking search) a step length which satisfies some minimum 'progress' conditions, the most common of which are called the Wolfe conditions. For our problems only one such condition is necessary and it is:

$$M(f_k + \alpha_k p_k) \le M(f_k) + c_1 \alpha_k \nabla M_k^T p_k,$$
(30)

where  $0 < c_1 < 1$  [5]. This condition guarantees that we will find a satisfactory step length for each iteration.

To determine the step direction we use the modified Gauss-Newton step:

$$\vec{p}_k = -(J_k^T J_k + \lambda I)^{-1} J_k^T \vec{F}(f_k),$$
 (31)

where  $J_k$  is the Jacobian of  $\tilde{f}_k$  and  $\lambda_k$  is called the forcing term. The forcing term is needed because during our iterations it is possible that we end up near a local minimum where  $\nabla M(f_k) = 0$ , but  $M(\tilde{f}^*) \neq 0$ . This would cause the line search to give  $\alpha_k = 0$ , and cause  $J_k^T J_k$  to become singular, which in turn causes the classic Gauss-Newton step to become undefined. Modifying the GN by adding the forcing term prevents this by ensuring  $(J_k^T J_k + \lambda I)$  is not singular. This also allows our line search to always give us a finite step length.

Choosing the value for  $\lambda_k$  is not trivial. If the value is too high, then the step direction becomes closer to the steepest decent direction (since as  $\lambda_k \to \infty$ ,  $p_k \to -J_k^T \vec{F}(f_k)$ ), and fast convergence is lost. If the value for  $\lambda_k$  is too small, then near non-zero roots of M, the length of each step becomes very small, and fast convergence is lost. Through experimentation, we find that a fixed value of  $\lambda_k = 0.0001$  works well for finding steadystate vortex profiles with our chosen parameters [5].

Our stopping criterion is when  $M(\vec{f}) < \epsilon_{\rm GM}$ , where  $\epsilon_{\rm GM}$  is our tolerance. Typically we set this to be between  $10^{-8}$  and  $10^{-4}$ .

## 4.2 Numerical Steady-State Profile Results

In Fig. 7, we show some radial profiles computed with our GN routine for various values of  $\Omega$  with charges m = 1 and m = 5. We show a couple of profiles with  $\Omega > 0.18$  including one with a very large  $\Omega = 0.1874$ . these results are consistent with the existence bound of  $\Omega_{max}^{2D} = 0.1875$  reported in [6] and predicted by our variational approach.



Figure 7: Steady-state vortex radial profiles computed with the GN routine for various values of  $\Omega$ . Left: m = 1, Right: m = 5. The top profiles have a range of  $Omega \in [0.01, 0.17]$  with  $\Delta \Omega = 0.01$ , while the bottom profiles have  $\Omega = [0.170.180.1870.1874]$ . For all profiles we used our VA ansatz as an initial condition. We used  $\Delta r = 0.5$  and a tolerance of  $10^-4$  in the GN routine.

To confirm that the profile solutions we compute from the GN routine are true steady-state vortex solutions, we use them as initial conditions to a full two-dimensional simulation of the CQNLS and run them for a long time and we see that they do appear to be steady-state.

#### 5 Conclusion

We have found vortex solutions to the two dimensional Cubic-Quintic Nonlinear Schrödinger Equation (CQNLS) for various values of the complex frequency. We have formulated a VA ansatz for the radial profile of the vortex which is very close to teh true solution. By using an optimization routine, we were able to refine the VA ansatz into numerically-exact vortex profiles. In so doing, we have confirmed the existence bounds of the vortices, that they are the same as the one dimensional existence bounds. These results will be used in further research to study the azimuthal modulational stability of vortices in the CQNLS.

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