Computational Science Research Center (CSRC)
and
Aerospace Engineering and Mechanics Seminars
San Diego State University
San Diego, CA

March 25, 2011
3:30pm-4:30pm

z-Transform Methods for the Optimal Design and Resonance Frequencies of a Finite Layered Elastic Strip

Ani P. Velo *(Presenter)*
Department of Mathematics and Computer Science
University of San Diego (USD), California

George A. Gazonas
U.S. Army Research Laboratory
Aberdeen Proving Ground, Maryland

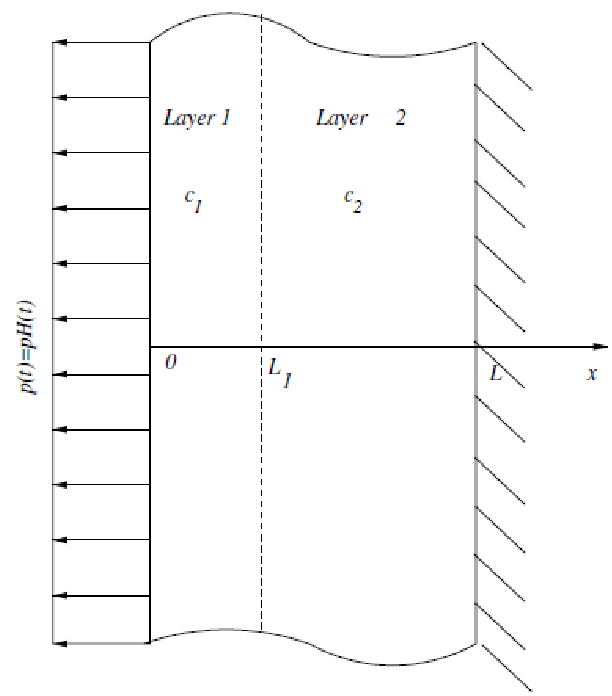
Outline

- Problem Description, Motivation, and Prior Work
- Results
  - Stress Wave Propagation: Formulation and Solution of Recursive Stress Formulas using the z-Transform Method
  - Optimization: Optimal Layered Designs with the Smallest Stress Amplitude
    - Applications of our Findings
  - Resonance and Natural Frequency Spectrum
    - Applications of our Findings
- Summary and Future Work

Problem Description

- 1 dimensional Layered Elastic Media
  Convenient for analytical calculations.
- B.C.: Discrete forcing function at one end and held fixed at the other end
  - *Heaviside loading when Minimizing the Stress Amplitude*
  - *Harmonic loading* when studying *Resonance*
- Goupillaud-type, i.e. equal wave travel time for each layer
  The stress wave/jump discontinuities meet and split at the layer interfaces, in addition to the boundary.
- Continuity of stress and displacement at each layer interface.

Two Layered Strip subjected to Heaviside Loading

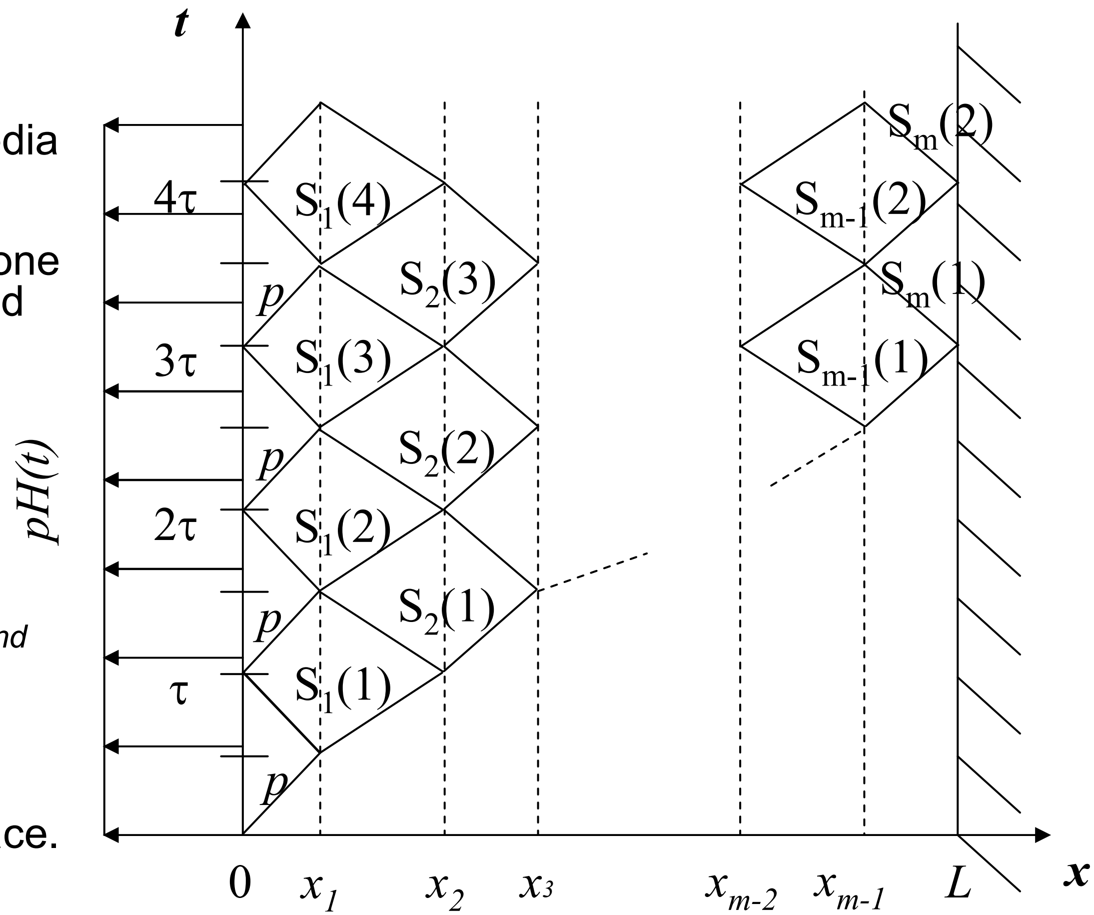


- Goal: Obtain analytical solutions for the stress propagation with the purpose to:
  - Minimize Stress Amplitude for a Heaviside Loading.
  - Generate Resonance Frequency Spectrum for a Harmonic Loading.

Problem Description

- 1 dimensional Layered Elastic Media
  Convenient for analytical calculations.
- B.C.: Discrete forcing function at one end and held fixed at the other end
  - *Heaviside loading when Minimizing the Stress Amplitude*
  - *Harmonic loading* when studying *Resonance*
- Goupillaud-type, i.e. equal wave travel time for each layer
  The stress wave/jump discontinuities meet and split at the layer interfaces, in addition to the boundary.
- Continuity of stress and displacement at each layer interface.

Lagrangian Diagram for a Heaviside Loading



- Goal: Obtain analytical solutions for the stress propagation with the purpose to:
  - Minimize Stress Amplitude for a Heaviside Loading.
  - Generate Resonance Frequency Spectrum for a Harmonic Loading.

Problem Motivation

- Despite the widespread use of multilayered structures in various technologies, exact analytical solutions to optimal design problems governed by the wave equation are rare in the literature. Such solutions are especially useful for testing optimization codes.

- The study of natural vibrations in elastic media include the study of resonance, as resonance can enhance the performance of many sensors and devices, yet can devastate structures subjected to sustained temporally-periodic loading, for instance during earthquakes.

Brief Literature Review on Optimization

- **[Anfinsen (1967)]** investigates the problem of maximizing or minimizing the amplitude of stress waves propagating through a finite bar of two layers, subjected to a transient stress loading. He determines optimal elastic properties of the structure by means of a finite difference method.

- **[Lee et al., Chiu & Erdogan]**, and others consider problems related to the propagation of stress waves in layered media. However they do not address the problem of optimization of inhomogeneous transiently loaded media:

  – **[Lee et al., (1975)]** provide error estimates for the stress, when a medium with continuous property variation is replaced by a medium consisting of a series of discrete homogeneous layers.

  – **[Chiu & Erdogan (1999)]** solve several transient wave-propagation boundary value problems for fixed/free and free/free end conditions in power-law one-dimensional FGM using Laplace transforms.

Brief Literature Review on Resonance

- **[S. D. Poisson, 1828]** *On the vibrations of an elastic sphere* -- the first to determine the free radial vibrations of a homogeneous sphere.
- **[H. Lamb, 1882]** *On the vibrations of an elastic sphere* -- calculated some of its natural frequencies.
- **[A. E. H. Love, 2002]** *A treatise on the mathematical theory of elasticity* -- provides an historical account of the early developments in this field.
- Literature includes the study of resonance in anisotropic elastic bodies, anisotropic layered crystals, elastic plates, periodic media, laminated and sandwich plates, composite laminates, piezoelectric composites, etc.
- Despite the long history of developments in the field, exact solutions for the resonance response of multilayered elastic media have been primarily limited to analyses involving only a few layers.

Further Simplifications of our Problem
m-Layered Media

- Transformation of the spatial variable

$$\xi = \int_0^x \frac{ds}{c(s)},$$

simplifies the problem to:

  - Goupillaud-type media with equal layer lengths $\frac{\tau}{m}$,
  - equal travel time of $\frac{\tau}{m}$ for each layer in either direction,
  - wave speed of unity.

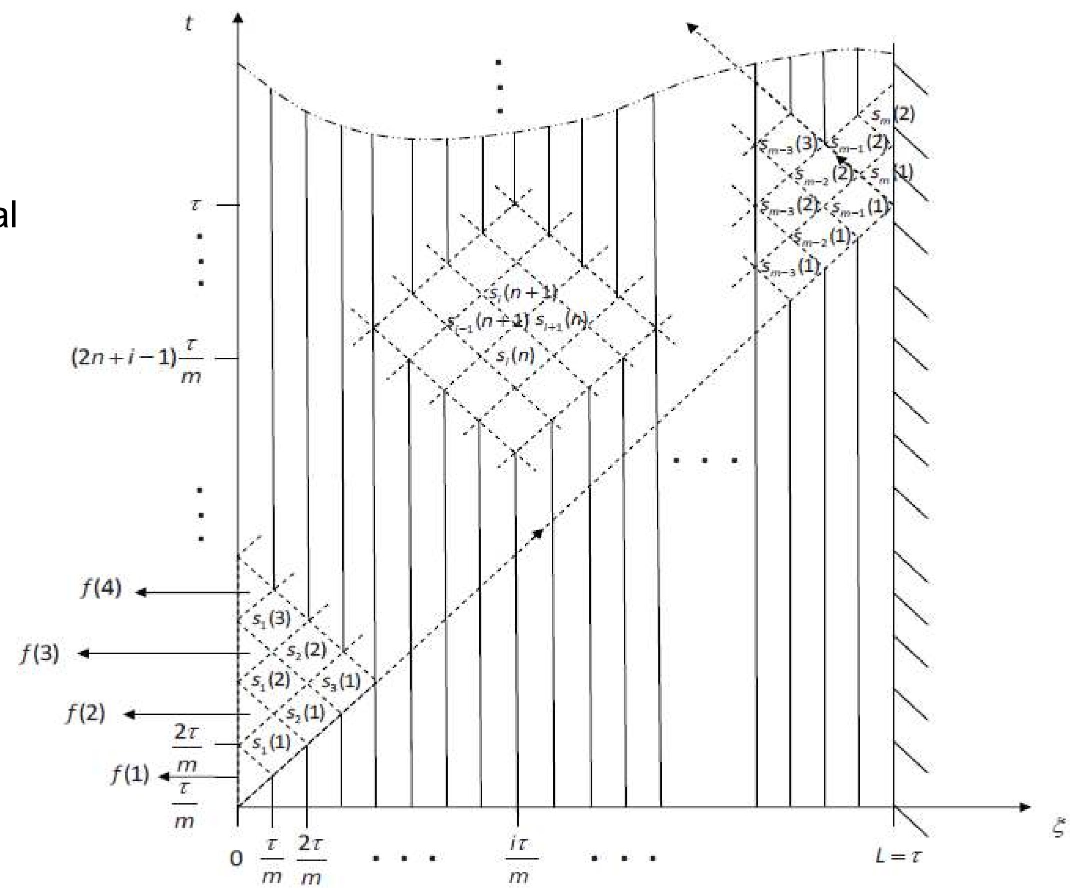
- The density and elastic modulus change as a result of this transformation (marked with ~).

$$\tilde{\rho}_i = \tilde{E}_i = \frac{E_i}{c_i} = \sqrt{E_i \rho_i}$$

- The impedance ratios of two consecutive layers and the stress values remain the same as a result of this transformation.

$$\alpha_i = \frac{\sqrt{E_i \rho_i}}{\sqrt{E_{i+1}\rho_{i+1}}} = \frac{\sqrt{\tilde{E}_i \tilde{\rho}_i}}{\sqrt{\tilde{E}_{i+1}\tilde{\rho}_{i+1}}} = \frac{\tilde{E}_i}{\tilde{E}_{i+1}}$$

$$E_1 \frac{\partial u}{\partial x} = \frac{E_1}{c_1}\frac{\partial u}{\partial \xi} = \tilde{E}\frac{\partial u}{\partial \xi}$$

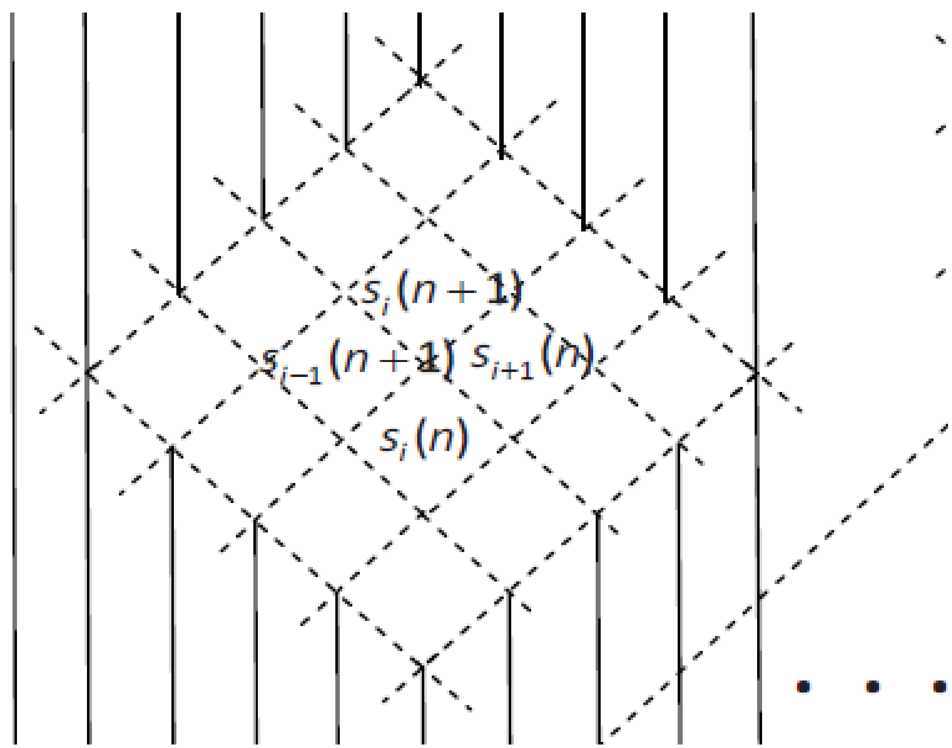


Lagrangian Diagram for a Discrete Loading

Discrete Forcing Function
Exact Difference Scheme for the Stress Terms

- **[K. Bube and R. Burridge, 1983]** *The one dimensional inverse problem of reflection seismology.*

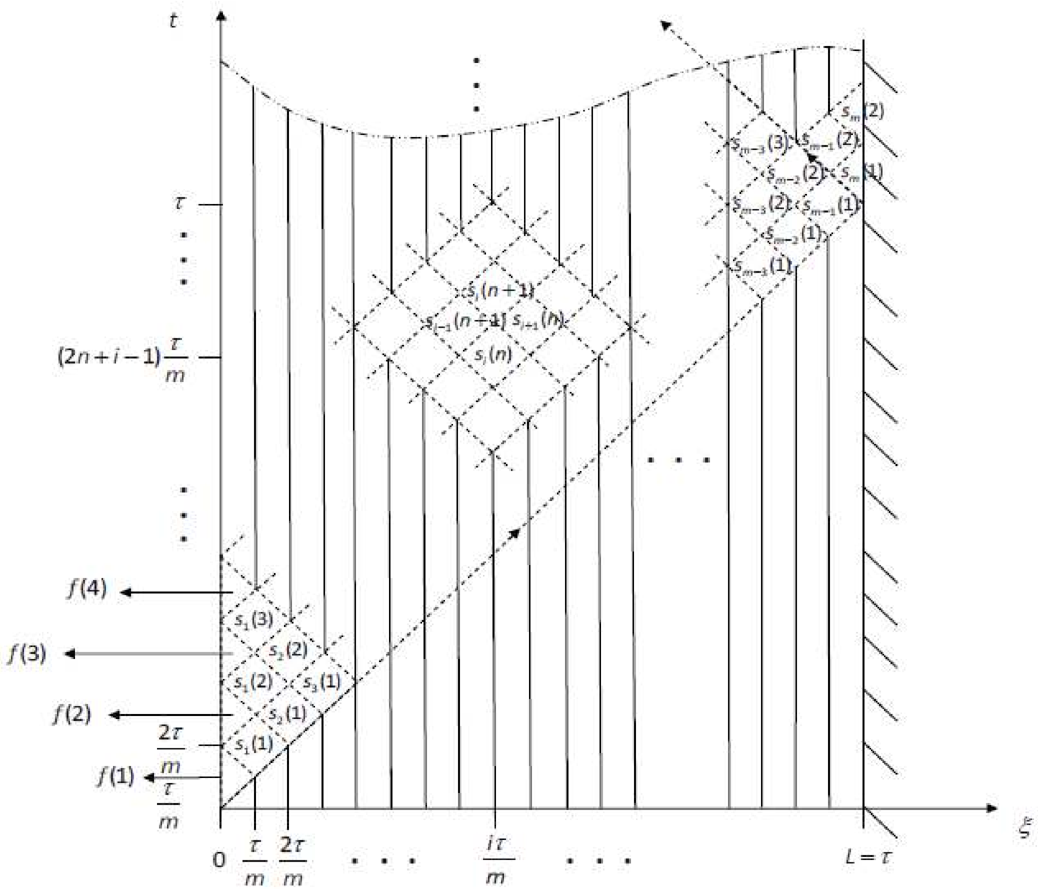
$$\boxed{s_i(n+1) + s_i(n) = \frac{2\alpha_i}{1+\alpha_i} s_{i+1}(n) + \frac{2}{1+\alpha_i} s_{i-1}(n+1)}$$



Discrete Forcing Function
Exact Difference Scheme for the Stress Terms

Recursive Relations:

$$\begin{cases} s_1(n+1) = -s_1(n) + \frac{2\alpha_1}{1+\alpha_1} s_2(n) + \frac{2}{1+\alpha_1} f(n+1), \\ s_i(n+1) = -s_i(n) + \frac{2\alpha_i}{1+\alpha_i} s_{i+1}(n) + \frac{2}{1+\alpha_i} s_{i-1}(n+1), \quad \text{for } \ i = 2, \dots, m-1, \\ s_m(n+1) = -s_m(n) + 2s_{m-1}(n+1), \end{cases}$$



z-Transform Approach

- Apply *z*-Transform to solve the system of recursive relations

$$\begin{cases} zS_1(z) = -S_1(z) + \frac{2\alpha_1}{1+\alpha_1} S_2(z) + \frac{2}{1+\alpha_1} z(F(z) - f(0)), \\ zS_i(z) = -S_i(z) + \frac{2\alpha_i}{1+\alpha_i} S_{i+1}(z) + \frac{2}{1+\alpha_i} zS_{i-1}(z), & \text{for } \; i = 2, \dots, m-1, \\ zS_m(z) = -S_m(z) + 2zS_{m-1}(z). \end{cases}$$

- Obtain a Linear System $\boxed{A_m \vec{x}_m = \vec{b}_m}$ with a tri-diagonal system matrix:

$$A_m = \begin{bmatrix} z+1 & -\eta_1\alpha_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -\eta_2 z & z+1 & -\eta_2\alpha_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\eta_3 z & z+1 & -\eta_3\alpha_3 & 0 & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & \cdots & 0 & -\eta_{m-1} z & z+1 & -\eta_{m-1}\alpha_{m-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & -\eta_m z & z+1 \end{bmatrix}_{m\times m}$$

$$\vec{x}_m = \begin{bmatrix} S_1(z) \\ S_2(z) \\ \vdots \\ S_m(z) \end{bmatrix}_{m\times 1}$$

$$\vec{b}_m = \begin{bmatrix} \eta_1 z(F(z) - f(0)) \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$$

$\eta_i = \dfrac{2}{1+\alpha_i}$ for $i = 1, \dots, m-1, \eta_m = 2.$

About the Determinant of the System Matrix

- The determinant of the system matrix is a palindromic polynomial

  Property: For an even power palindromic polynomial the roots come in inverse pairs.

$$
|A_m| = \begin{cases} z^m + a_{m,1}z^{m-1} + a_{m,2}z^{m-2} + \cdots + a_{m,\frac{m}{2}-1}z^{\frac{m}{2}+1} + a_{m,\frac{m}{2}}z^{\frac{m}{2}} + \\ +a_{m,\frac{m}{2}-1}z^{\frac{m}{2}-1} + \cdots + a_{m,2}z^2 + a_{m,1}z + 1, & \text{for } m\text{-even}, \\ \\ (z+1)[z^{m-1} + b_{m,1}z^{m-2} + \cdots + b_{m,\frac{m-1}{2}-1}z^{\frac{m-1}{2}+1} + b_{m,\frac{m-1}{2}}z^{\frac{m-1}{2}} + \\ +b_{m,\frac{m-1}{2}-1}z^{\frac{m-1}{2}-1} + \cdots + b_{m,2}z^2 + b_{m,1}z + 1], & \text{for } m\text{-odd}. \end{cases}
$$

$$|A_1| = z + 1,$$

$$|A_2| = (z+1)^2 - \eta_2\eta_1\alpha_1 z = z^2 - \frac{2(\chi_1 - 2)}{\chi_1}z + 1,$$

$$|A_3| = (z+1)\cdot [\, z^2 - \frac{2(\chi_2-2)}{\chi_2}z + 1 \,],$$

$$|A_4| = z^4 - \frac{4\Gamma_3}{\chi_3}z^3 + \frac{2(4\Gamma_3 - \chi_3 + 8)}{\chi_3}z^2 - \frac{4\Gamma_3}{\chi_3}z + 1,$$

$$|A_5| = (z+1)\cdot [\, z^4 - \frac{4\Gamma_4}{\chi_4}z^3 + \frac{2(4\Gamma_4 - \chi_4 + 8)}{\chi_4}z^2 - \frac{4\Gamma_4}{\chi_4}z + 1 \,].$$

design parameters are given by $\chi_{m-1} = \prod_{i=1}^{m-1}(1+\alpha_i)$ for $2 \le m \le 5$, $\Gamma_3 = \alpha_1\alpha_3 - 1$, and $\Gamma_4 = \alpha_1\alpha_3\alpha_4 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_4 + \alpha_2\alpha_4 + \alpha_1\alpha_3 - 1$. Here $\chi_{m-1} > 1$ for $m \ge 2$ and $\Gamma_{m-1} > -1$ for $m = 4, 5$.

About the Determinant of the System Matrix

- Proved that the roots of the determinant are distinct and lie on the unit circle for up to five layers.
- Using tri-diagonal Toeplitz matrices, we can find designs for which the roots of the determinant are distinct and lie on the unit circle for any number of layers.
- We consider designs for which the roots of the determinant are distinct and lie on the unit circle. The roots of the even power polynomial come in complex conjugate pairs,

$$z_k + z_k^{-1} = z_k + \bar{z_k} = 2\cos\theta_k$$

and the determinant can be factored as:

$$|A_m| = \begin{cases} \prod_{k=1}^{\lfloor \frac{m}{2} \rfloor} [z^2 - 2z\cos\theta_k + 1] & \text{for } m \text{ even,} \\ (z+1) \prod_{k=1}^{\lfloor \frac{m}{2} \rfloor} [z^2 - 2z\cos\theta_k + 1] & \text{for } m \text{ odd.} \end{cases}$$

The angles $\{\pm\theta_k\}_{k=1}^{\frac{m}{2}}$ correspond to the roots $z = e^{\pm \mathcal{I}\theta_k}$ for m even, while the angles $\theta_0 = \pi$, $\{\pm\theta_k\}_{k=1}^{\lfloor \frac{m}{2} \rfloor}$ correspond to the roots $z = e^{\mathcal{I}\theta_0} = -1$ and $z = e^{\pm \mathcal{I}\theta_k}$ respectively for m odd.

Solving the Stress Recursive Relations

$$\vec{x}_m = A_m^{-1} \vec{b}_m = \frac{\eta_1 p z}{(z-1)|A_m|} \begin{pmatrix} |A_{1,1}| \\ -|A_{1,2}| \\ \vdots \\ (-1)^m |A_{1,m}| \end{pmatrix}$$

here $A_{i,j}$ is a minor of A_m

Partial Fraction

Inverse z-Transform

Recover $s_i(n)$

Explicit Stress Formulas for a Discrete Loading

- Obtain stress solutions, after applying the inverse *z*-transform:

$$s_i(n) = Z^{-1}(\vec{x}_m(i)) = \quad f(n) * \left[b_{i,0}(-1)^n + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} a_{i,k}\cos(n\theta_k) + b_{i,k}\sin(n\theta_k) \right]$$

$$-f(0) \cdot \left[b_{i,0}(-1)^n + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} a_{i,k}\cos(n\theta_k) + b_{i,k}\sin(n\theta_k) \right]$$

- Special Case I: Heaviside Loading

$$f(n) \;=\; p \;\text{ for }\; n \;\geq\; 0$$

$$s_i(n) = \left[\; a_{i,0} + b_{i,0}(-1)^n + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} a_{i,k}\cos(n\theta_k) + b_{i,k}\sin(n\theta_k) \right] \cdot p$$

- Special Case II: Discrete Harmonic Loading

$$f(n) = \sin(n\tilde{\omega}),\; n \geq 0$$

- Special Case I: Heaviside Loading and Optimization

$$f(n) = p \text{ for } n \geq 0$$

$$s_i(n) = \left[a_{i,0} + b_{i,0}(-1)^n + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} a_{i,k} \cos(n\theta_k) + b_{i,k} \sin(n\theta_k) \right] \cdot p$$

Stress Formulas for the Two-Layer Case
Heaviside Loading

$$\begin{cases} s_1(n) = [\ 1 - \cos n\theta_1\ ] \cdot p, \\ \\ s_2(n) = [\ 1 - \cos n\theta_1 - \frac{1}{\sqrt{\chi_1 - 1}} \sin n\theta_1\ ] \cdot p, \end{cases} \qquad \text{for } n \geq 1.$$

Here the design parameter χ_1 is given by $\chi_1 = (\alpha_1 + 1)$

For $\chi_1 > 2$ and $0 < \theta_1 < \frac{\pi}{2}$, we have that

$$\boxed{\chi_1 = \frac{2}{1 - \cos\theta_1} \quad \text{or, equivalently,} \quad \cos\theta_1 = \frac{\chi_1 - 2}{\chi_1} = \frac{\alpha_1 - 1}{\alpha_1 + 1}.}$$

Stress Formulas for the Three-Layer Case
Heaviside Loading

$$\begin{cases} s_1(n) = \left[\, 1 - \frac{\alpha_2}{\chi_2 - 1} \; (-1)^n - \frac{\alpha_1 + \alpha_1\alpha_2}{\chi_2 - 1} \; \cos n\theta_1 \right] \cdot p, \\ s_2(n) = \left[\, 1 - \cos n\theta_1 + \frac{1}{\sqrt{\chi_2 - 1}} \sin n\theta_1 \right] \cdot p, \qquad\qquad\qquad\qquad \text{for } n \geq 1. \\ s_3(n) = \left[\, 1 - \frac{1}{\chi_2 - 1} \; (-1)^n - \frac{\chi_2 - 2}{\chi_2 - 1} \cos n\theta_1 + \frac{2}{\sqrt{\chi_2 - 1}} \sin n\theta_1 \right] \cdot p, \end{cases}$$

Here the design parameter χ_2 is given by $\chi_2 = (\alpha_1 + 1)(\alpha_2 + 1) > 1$,

for $\chi_2 > 2$ and $0 < \theta_1 < \frac{\pi}{2}$, we have that

$$\boxed{\chi_2 = \frac{2}{1 - \cos\theta_1} \quad \text{or, equivalently,} \quad \cos\theta_1 = \frac{\chi_2 - 2}{\chi_2}.}$$

Stress Formulas for the Four-Layer Case – Maple results
Heaviside Loading

$$
\begin{cases}
s_1(n) = a_{1,0} + a_{1,1}\cos n\theta_1 + b_{1,1}\sin n\theta_1 + a_{1,2}\cos n\theta_2 + b_{1,2}\sin n\theta_2 \\
s_2(n) = a_{2,0} + a_{2,1}\cos n\theta_1 + b_{2,1}\sin n\theta_1 + a_{2,2}\cos n\theta_2 + b_{2,2}\sin n\theta_2 \\
s_3(n) = a_{3,0} + a_{3,1}\cos n\theta_1 + b_{3,1}\sin n\theta_1 + a_{3,2}\cos n\theta_2 + b_{3,2}\sin n\theta_2 \\
s_4(n) = a_{4,0} + a_{4,1}\cos n\theta_1 + b_{4,1}\sin n\theta_1 + a_{4,2}\cos n\theta_2 + b_{4,2}\sin n\theta_2.
\end{cases}
$$

where:

$$
\cos\theta_1 = \frac{\eta_3\alpha_3}{2} + \frac{\eta_3\eta_2\alpha_2}{4} + \frac{\eta_2\eta_1\alpha_1}{4} - 1 \pm \frac{\sqrt{4\,\eta_3^{\,2}\alpha_3^{\,2}+4\,\eta_3^{\,2}\alpha_3\eta_2\alpha_2-4\,\eta_2\eta_1\alpha_1\eta_3\alpha_3+\eta_3^{\,2}\eta_2^{\,2}\alpha_2^{\,2}+2\,\eta_3\eta_2^{\,2}\alpha_2\eta_1\alpha_1+\eta_2^{\,2}\eta_1^{\,2}\alpha_1^{\,2}}}{4}
$$

$$
\cos\theta_2 = \frac{\eta_3\alpha_3}{2} + \frac{\eta_3\eta_2\alpha_2}{4} + \frac{\eta_2\eta_1\alpha_1}{4} - 1 \mp \frac{\sqrt{4\,\eta_3^{\,2}\alpha_3^{\,2}+4\,\eta_3^{\,2}\alpha_3\eta_2\alpha_2-4\,\eta_2\eta_1\alpha_1\eta_3\alpha_3+\eta_3^{\,2}\eta_2^{\,2}\alpha_2^{\,2}+2\,\eta_3\eta_2^{\,2}\alpha_2\eta_1\alpha_1+\eta_2^{\,2}\eta_1^{\,2}\alpha_1^{\,2}}}{4}
$$

$$
\begin{aligned}
a_{1,0} &= \frac{1}{2}\,\frac{p\eta_1(\eta_3\eta_2\alpha_2+2\,\eta_3\alpha_3-4)}{(-1+\cos\theta_1)(-1+\cos\theta_2)}, && \\
a_{1,1} &= \frac{1}{2}\,\frac{\eta_1 p(2-\eta_3\eta_2\alpha_2-2\,\eta_3\alpha_3+2\cos\theta_1)}{-\cos\theta_2+\cos\theta_1\cos\theta_2-\cos\theta_1^{\,2}+\cos\theta_1}, & b_{1,1} &= 0, \\
a_{1,2} &= -\frac{1}{2}\,\frac{(-\eta_3\eta_2\alpha_2-2\,\eta_3\alpha_3+2\cos\theta_2+2)\eta_1 p}{(-\cos\theta_1+\cos\theta_2)(-1+\cos\theta_2)}, & b_{1,2} &= 0,
\end{aligned}
$$

$$
\begin{aligned}
a_{2,0} &= -\frac{1}{2}\,\frac{\eta_2 p\eta_1(-2+\eta_3\alpha_3)}{(-1+\cos\theta_1)(-1+\cos\theta_2)}, && \\
a_{2,1} &= -\frac{1}{2}\,\frac{\eta_1 p\eta_2(1-\eta_3\alpha_3+\cos\theta_1)}{-\cos\theta_2+\cos\theta_1\cos\theta_2-\cos\theta_1^{\,2}+\cos\theta_1}, & b_{2,1} &= -\frac{1}{2}\,\frac{\eta_1 p\eta_2(1-\eta_3\alpha_3+\cos\theta_1)}{(-\cos\theta_1+\cos\theta_2)s1}, \\
a_{2,2} &= \frac{1}{2}\,\frac{(-\eta_3\alpha_3+\cos\theta_2+1)\eta_1 p\eta_2}{(-\cos\theta_1+\cos\theta_2)(-1+\cos\theta_2)}, & b_{2,2} &= \frac{1}{2}\,\frac{(-\eta_3\alpha_3+\cos\theta_2+1)\eta_1 p\eta_2}{s2\,(-\cos\theta_1+\cos\theta_2)},
\end{aligned}
$$

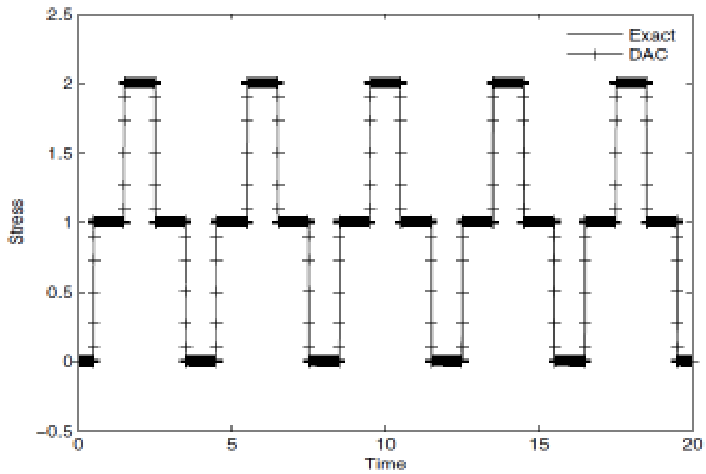
$$
\begin{aligned}
a_{3,0} &= -\frac{1}{2}\,\frac{\eta_2 p\eta_1(-2+\eta_3\alpha_3)}{(-1+\cos\theta_1)(-1+\cos\theta_2)}, && \\
a_{3,1} &= -\frac{1}{2}\,\frac{\eta_1 p\eta_2(1-\eta_3\alpha_3+\cos\theta_1)}{-\cos\theta_2+\cos\theta_1\cos\theta_2-\cos\theta_1^{\,2}+\cos\theta_1}, & b_{3,1} &= -\frac{1}{2}\,\frac{\eta_1 p\eta_2(1-\eta_3\alpha_3+\cos\theta_1)}{(-\cos\theta_1+\cos\theta_2)s1}, \\
a_{3,2} &= \frac{1}{2}\,\frac{(-\eta_3\alpha_3+\cos\theta_2+1)\eta_1 p\eta_2}{(-\cos\theta_1+\cos\theta_2)(-1+\cos\theta_2)}, & b_{3,2} &= \frac{1}{2}\,\frac{(-\eta_3\alpha_3+\cos\theta_2+1)\eta_1 p\eta_2}{s2\,(-\cos\theta_1+\cos\theta_2)}
\end{aligned}
$$

$$
\begin{aligned}
a_{4,0} &= \frac{1}{2}\,\frac{\eta_1 p\eta_3\eta_2}{(-1+\cos\theta_1)(-1+\cos\theta_2)}, && \\
a_{4,1} &= -\frac{1}{2}\,\frac{\eta_1 p\eta_3\eta_2(-1+2\cos\theta_1)}{-\cos\theta_2+\cos\theta_1\cos\theta_2-\cos\theta_1^{\,2}+\cos\theta_1}, & b_{4,1} &= -\frac{1}{2}\,\frac{(1+2\cos\theta_1)\eta_1 p\eta_3\eta_2}{(-\cos\theta_1+\cos\theta_2)s1}, \\
a_{4,2} &= \frac{1}{2}\,\frac{(-1+2\cos\theta_2)\eta_1 p\eta_3\eta_2}{(-\cos\theta_1+\cos\theta_2)(-1+\cos\theta_2)}, & b_{4,2} &= \frac{1}{2}\,\frac{\eta_1 p\eta_3\eta_2(2\cos\theta_2+1)}{s2\,(-\cos\theta_1+\cos\theta_2)}.
\end{aligned}
$$

Optimal Design Problem Formulation
Heaviside Loading, *m*-layer Case

$$\inf_{\vec{\eta}_m} \sup_{1 \leq n < +\infty} [p \ , s_1(n, p, \vec{\eta}_m), s_2(n, p, \vec{\eta}_m), \ldots, s_m(n, p, \vec{\eta}_m)]$$

where $\vec{\eta}_m = (\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$



Stress Time History at the middle of a homogeneous elastic strip, with a stress amplitude of double the loading p=1.

$$p \leq \inf_{\vec{\eta}_m} \sup_{1 \leq n < +\infty} [p \ , s_1(n, p, \vec{\eta}_m), s_2(n, p, \vec{\eta}_m), \ldots, s_m(n, p, \vec{\eta}_m)] \leq 2p.$$

$$\boxed{s_m(n = 1, \vec{\eta}_m, p) = \frac{2^m}{\chi_{m-1}} p}$$

$$\inf_{\vec{\eta}_m, \ \chi_{m-1} \geq 2^{m-1}} \sup_{1 \leq n < +\infty} [p, \ s_1(n, p, \vec{\eta}_m), s_2(n, p, \vec{\eta}_m), \ldots, s_m(n, p, \vec{\eta}_m)].$$

Stress Optimization for the Two-Layer Case
Heaviside Loading

$$\inf_{\chi_1 \geq 2} \sup_{n} \{S_1(n), S_2(n), p\} \leq 2p$$

- Bounds on the stress terms for any given α :

$$0 \leq \max_{n} S_1(n,\alpha) \leq 2p, \qquad \max_{n} S_2(n,\alpha) \geq 2p$$

- Optimality condition:

$$\min_{\chi_1 \geq 2} \max_{n} \{S_1(n), S_2(n), p\} = 2p$$

- Optimal values for the angle:

$$\boxed{\theta_{1,opt} = \frac{\pi}{j}, \quad j = 2, 3, 4, \ldots .}$$

and the design parameter:

$$\boxed{\chi_{1,opt} = \frac{2}{1 - \cos(\theta_{1,opt})} = \frac{2}{1 - \cos\frac{\pi}{j}}, \quad j = 2, 3, 4, \ldots ,}$$

Computationally-Derived Optimal Solutions for the Two-Layered Strip using DYNA3D/GLO

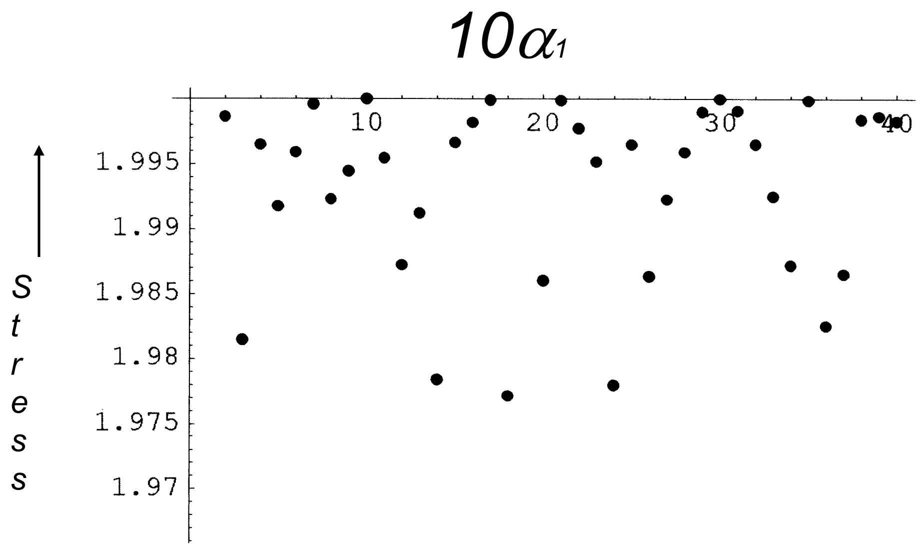
- Explicit Finite Element Code DYNA3D using 60 hexahedral finite elements.
- GLO – Global/Local Optimizer
  Optimal solutions obtained using a coupled global (discrete) and local (gradient, “variable metric”) method.
- Search limited to a region involving the first eight optimal design points.
- Comparison of analytical and DYNA3D/GLO derived $\chi_{1,opt}$ values.

j	2	3	4	5	6	7	8	9
$\chi_{1,opt}$	2	4	6.828	10.472	14.983	20.196	26.270	33.163
$\chi_{1,opt}$ DYNA/GLO	2	4	6.829	10.470	14.924	20.182	25.247	32.120

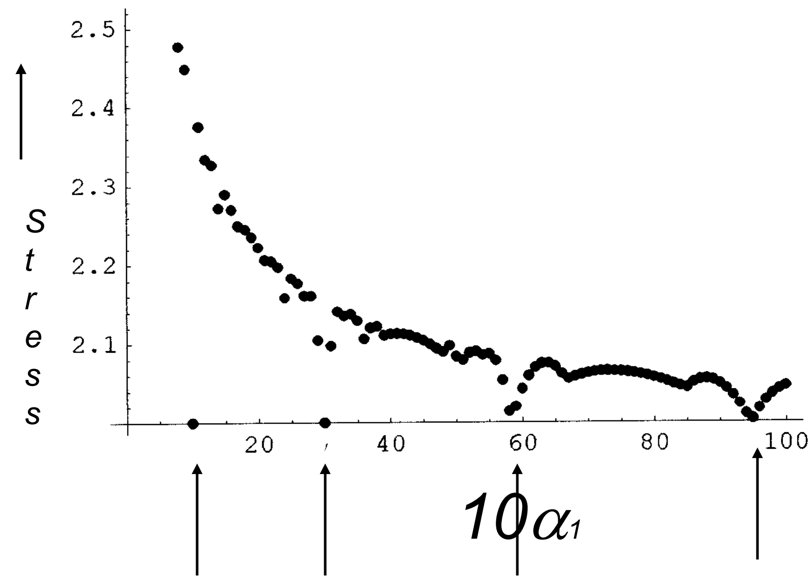
Optimal Design Verification for the Two-Layer Case

Maximum Stress versus design parameter/impedance ratio α_1.

Layer 1: 0 < x < ½



Layer 2: ½ < x < 1



$\chi_{1,opt}$ = 2 4 6.828 10.472

k= 2 3 4 5

Periodic Optimal Solutions for the Two-Layer Case

- Time Delay Benefit to reach Max Stress as j increases in value (j=3 first row vs. j=9 second row)

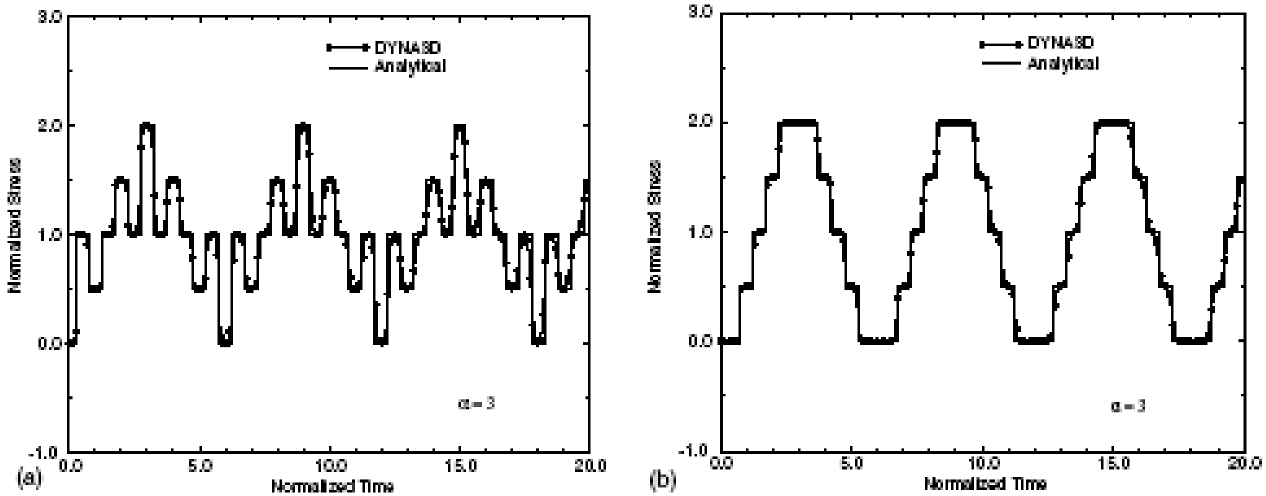


Fig. 4. Optimal stress history when $\tau = 1$ and $\alpha = 3$: (a) layer 1: $\xi = 0.25$, (b) layer 2: $\xi = 0.75$.

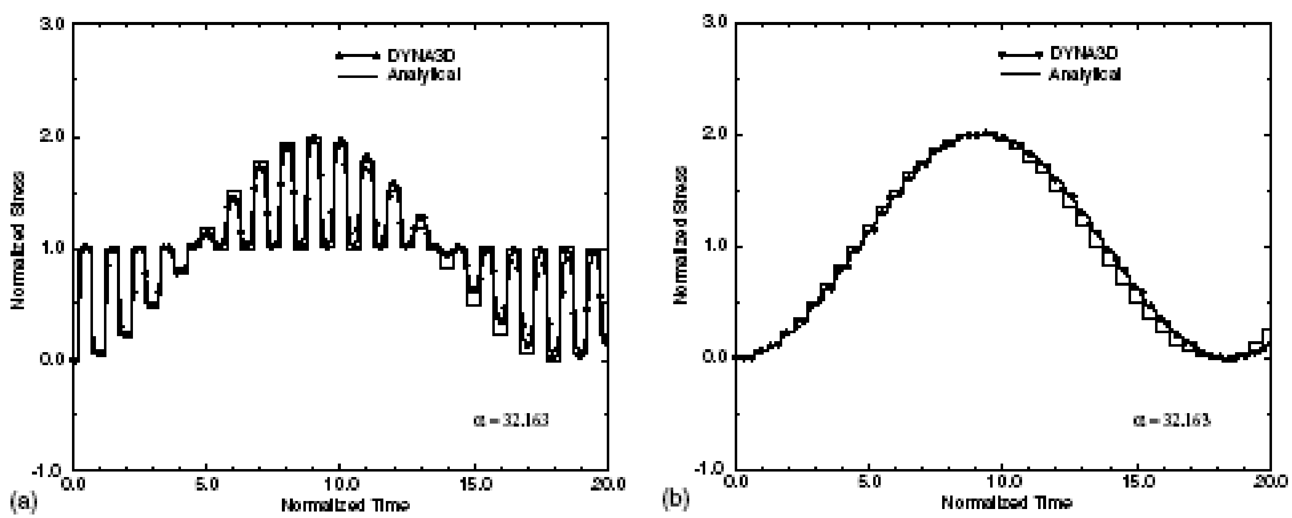


Fig. 5. Optimal stress history when $\tau = 1$ and $\alpha = 32.163$: (a) layer 1: $\xi = 0.25$, (b) layer 2: $\xi = 0.75$.

Stress Optimization for the Three-Layer Case
Heaviside Loading

- Optimal values for the angle:

$$\boxed{\theta_{1,opt} = \frac{\pi}{j}, \quad j = 3, 5, 7, \ldots,}$$

and the design parameter $\chi_2 = (1+\alpha_1)(1+\alpha_2)$

$$\boxed{\chi_{2,opt} = \frac{2}{1-\cos(\theta_{1,opt})} = \frac{2}{1-\cos\frac{\pi}{j}}, \quad j = 3, 5, 7, \ldots.}$$

j	3	5	7	9
$\chi_{2,opt}$	4	10.472	20.196	33.163

Stress Optimization for the Four-Layer Case
Heuristic Approach

- Optimality Condition too complicated to follow:

$$\boxed{\min_{\chi_3 \geq 8} \max_{n} \{S_1(n), S_2(n), S_3(n), S_4(n), p\} \leq 2p}$$

$$\chi_3 = (1+\alpha_1)(1+\alpha_2)(1+\alpha_3).$$

- Matching the powers of z for the two different representations of |Am| in the z-space:

$$\begin{aligned}|A_4| &= (z+1)|A_{\{1,2,3\}}| - \eta_4\alpha_3\eta_3 z|A_{\{1,2\}}| \\ &= z^4 - \frac{4\Gamma_3}{\chi_3}z^3 + \frac{2(4\Gamma_3 - \chi_3 + 8)}{\chi_3}z^2 - \frac{4\Gamma_3}{\chi_3}z + 1,\end{aligned}$$

$$\begin{aligned}|A_4| &= (z - e^{i\theta_1})(z - e^{-i\theta_1})(z + e^{i\theta_2})(z - e^{i\theta_2}) \\ &= (z^2 - 2z\cos\theta_1 + 1)(z^2 - 2z\cos\theta_2 + 1) \\ &= z^4 - 2(\cos\theta_1 + \cos\theta_2)z^3 + 2(1 + 2\cos\theta_1\cos\theta_2)z^2 - 2(\cos\theta_1 + \cos\theta_2)z + 1\end{aligned}$$

$$\Downarrow$$

$$\begin{cases}\cos\theta_1 + \cos\theta_2 = \frac{2\Gamma_3}{\chi_3}, \\ \cos\theta_1 \cdot \cos\theta_2 = \frac{2\Gamma_3 - \chi_3 + 4}{\chi_3}.\end{cases}$$

Here $\chi_3 = (1+\alpha_1)(1+\alpha_2)(1+\alpha_3)$ and $\Gamma_3 = (\alpha_1\alpha_3 - 1)$

Stress Optimization for the Four-Layer Case
Heuristic Approach

- Motivated from the homogeneous case, we apply the additional condition $\boxed{\Gamma_3 = \alpha_1\alpha_3 - 1 = 0}$ to the system:

$$\begin{cases} \cos\theta_1 + \cos\theta_2 = \frac{2\Gamma_3}{\chi_3}, \\ \cos\theta_1 \cdot \cos\theta_2 = \frac{2\Gamma_3 - \chi_3 + 4}{\chi_3}. \end{cases}$$

and obtain that: $\cos(\theta_1) = -\cos(\theta_2)$ and

$$\boxed{\chi_3 = (1+\alpha_1)(1+\alpha_2)(1+\alpha_3) = \frac{2^3}{1-\cos(2\theta_1)}.}$$

- Compare this with the optimal results for the 2-layer case $(\alpha_2 = \alpha, \alpha_1 = \alpha_3 = 1)$:

$$\boxed{\chi_{2,opt} = \frac{2}{1-\cos\frac{\pi}{k}},\ k = 2,3,4,\ldots\text{etc.}} \Rightarrow \boxed{\cos(2\theta_1) = \cos(\frac{\pi}{k})}$$

Optimal Design for the Four-Layer Case

- Proposed optimal values for the angle

$$\boxed{\theta_{1,opt} = \frac{\pi}{2j}, \quad \text{where } j = 2, 3, 4, \ldots,}$$

and design parameter:

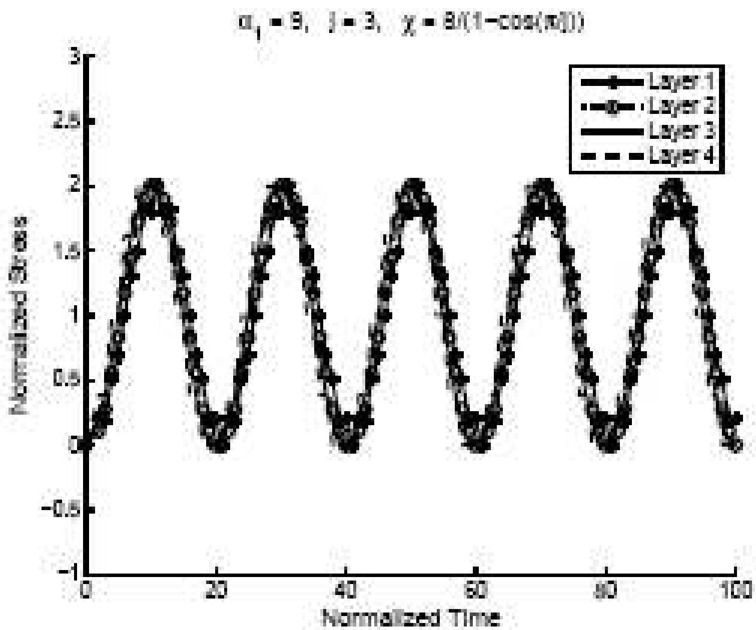
$$\begin{cases} \chi_{3,opt} = \frac{2^3}{1-cos(\frac{\pi}{j})}, & j = 2, 3, 4, \ldots, \\ \Gamma_{3,opt} = 0. \end{cases}$$

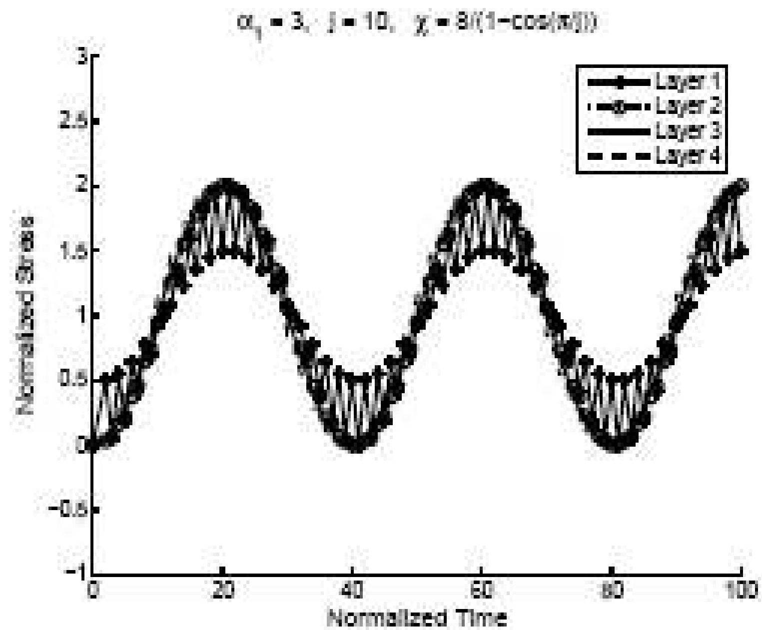
j	2	3	4	5	6	7	8	9	10
$\theta_{1,opt}$	$\pi/4$	$\pi/6$	$\pi/8$	$\pi/10$	$\pi/12$	$\pi/14$	$\pi/16$	$\pi/18$	$\pi/20$
$\chi_{3,opt}$	8	16	27.314	41.888	59.713	80.783	105.096	132.654	163.454

- Similar optimal results can be obtained for the five layer case.

Periodic Optimal Stress Solutions for the Four-Layer Case

- Optimal design stress time history:





Distribution of the Optimal Angle Values

2-Layer case

$$\theta_{1,opt} = \frac{\pi}{j}, \quad j = 2, 3, 4, \ldots.$$

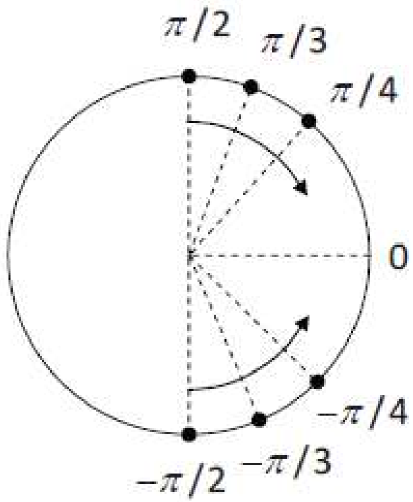
3-Layer case

$$\theta_{1,opt} = \frac{\pi}{j}, \quad j = 3, 5, 7, \ldots,$$

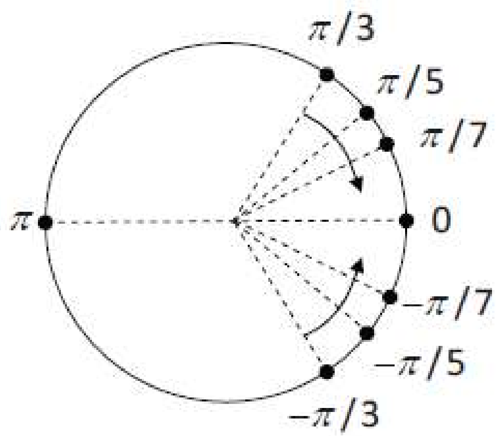
4-Layer case

$$\theta_{1,opt} = \frac{\pi}{2j}, \quad \text{where } j = 2, 3, 4, \ldots,$$

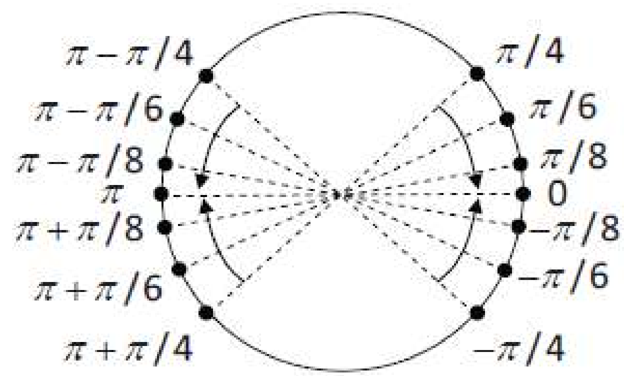
$$\cos(\theta_2) = -\cos(\theta_1)$$



m=2



m=3



m=4

Applications: Optimal Material Configurations and Design Improvement

- The two-layer case, $\chi_1 = 1 + \alpha$

  Homogeneous design:

  $$\chi_{1,opt} = 2 \Rightarrow \alpha = 1$$

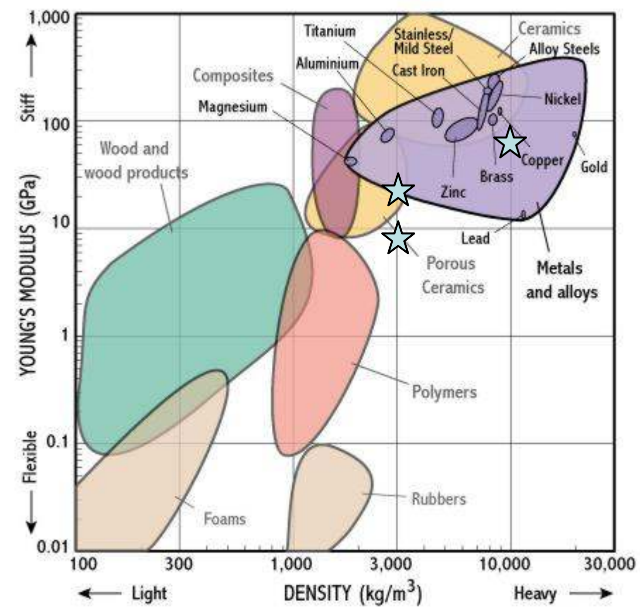
  Tungsten Alloy and Lead:

  $$\chi_{1,opt} = 6.828 \Rightarrow \alpha = 5.828$$

  Tungsten and Copper Alloy:

  $$\chi_{1,opt} = 33.163 \Rightarrow \alpha = 32.163$$

Material Selection Chart



- The Three-Layer Case, $\chi_2 = (1+\alpha_1)(1+\alpha_2)$.

  Homogeneous design:

  $$\alpha_1 = 1, \chi_{2,opt.} = 4 \Rightarrow \alpha_2 = 1$$

  Aluminum-Steel→Glass Fiber Reinforced

  $$\alpha_1 = .33, \chi_{2,opt.} = 10.472 \Rightarrow \alpha_2 = 6.87$$

- The Four-Layer Case, $\chi_3 = (1+\alpha_1)(1+\alpha_2)(1+\alpha_3)$

.

.Homogeneous design:

$$\alpha_1 = 1, \alpha_3 = \frac{1}{\alpha_1} = 1, \chi_{3,opt.} = 8 \Rightarrow \alpha_2 = 1$$

Non-Homogeneous Optimal Design:

$$\alpha_1 = 2.5, \alpha_3 = \frac{1}{\alpha_1} = 0.4, \chi_{3,opt.} = 8 \Rightarrow \alpha_2 = 0.633$$

Applications of our Optimality Results to Non-Goupillaud type Layered Media

- The optimality conditions for the non-Goupillaud type two-layered medium with wave travel time ratio 1:2 (or 2:1), can be derived from the Goupillaud-type three-layer medium with wave travel time ratio 1:1:1 and $\alpha_2=1$ (or $\alpha_1=1$).

$$\boxed{(1+\alpha_{opt}) = \frac{1}{1-\cos\dfrac{\pi}{(2k-1)}},\ k = 2,3,4...}$$

✓ Special Case I: Heaviside Loading and Optimization

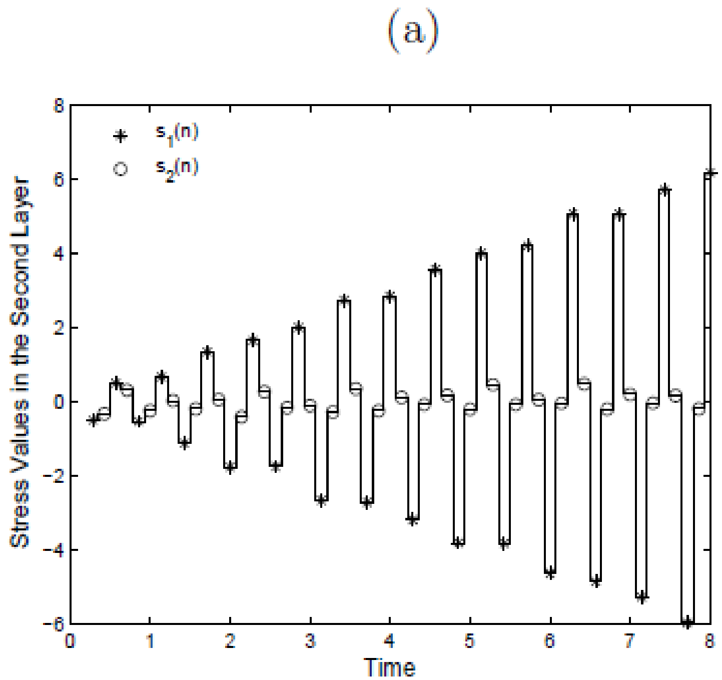
$$f(n) \ = \ p \ \text{for} \ n \ \geq \ 0$$

$$s_i(n) = \left[a_{i,0} + b_{i,0}(-1)^n + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} a_{i,k} \cos(n\theta_k) + b_{i,k} \sin(n\theta_k) \right] \cdot p$$

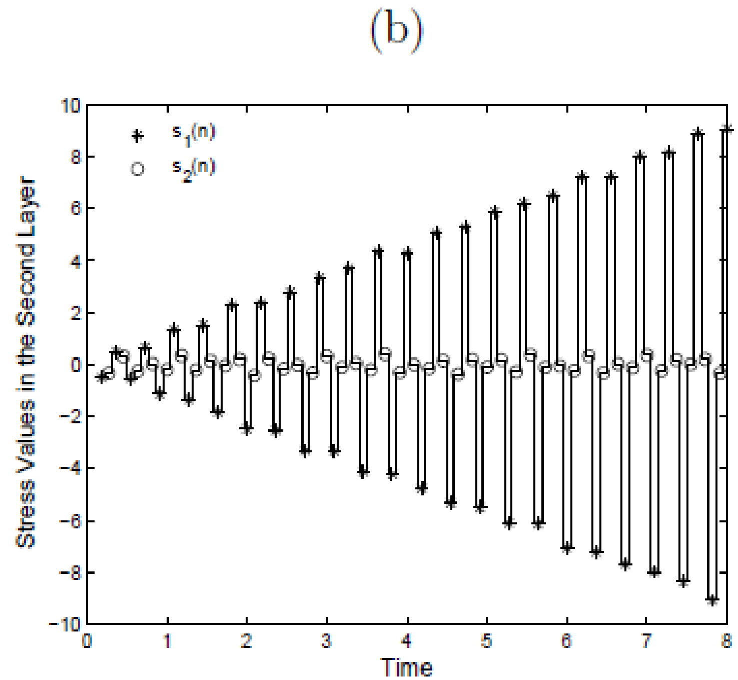
- Special Case II: Discrete Harmonic Loading and Resonance

$$f(n) = \sin(n\tilde{\omega}), \ n \geq 0$$

Numerical Experiments validating Universal Resonance Frequency for Multilayered Designs with an Odd Number of Layers.



(a) Seven-layered strip



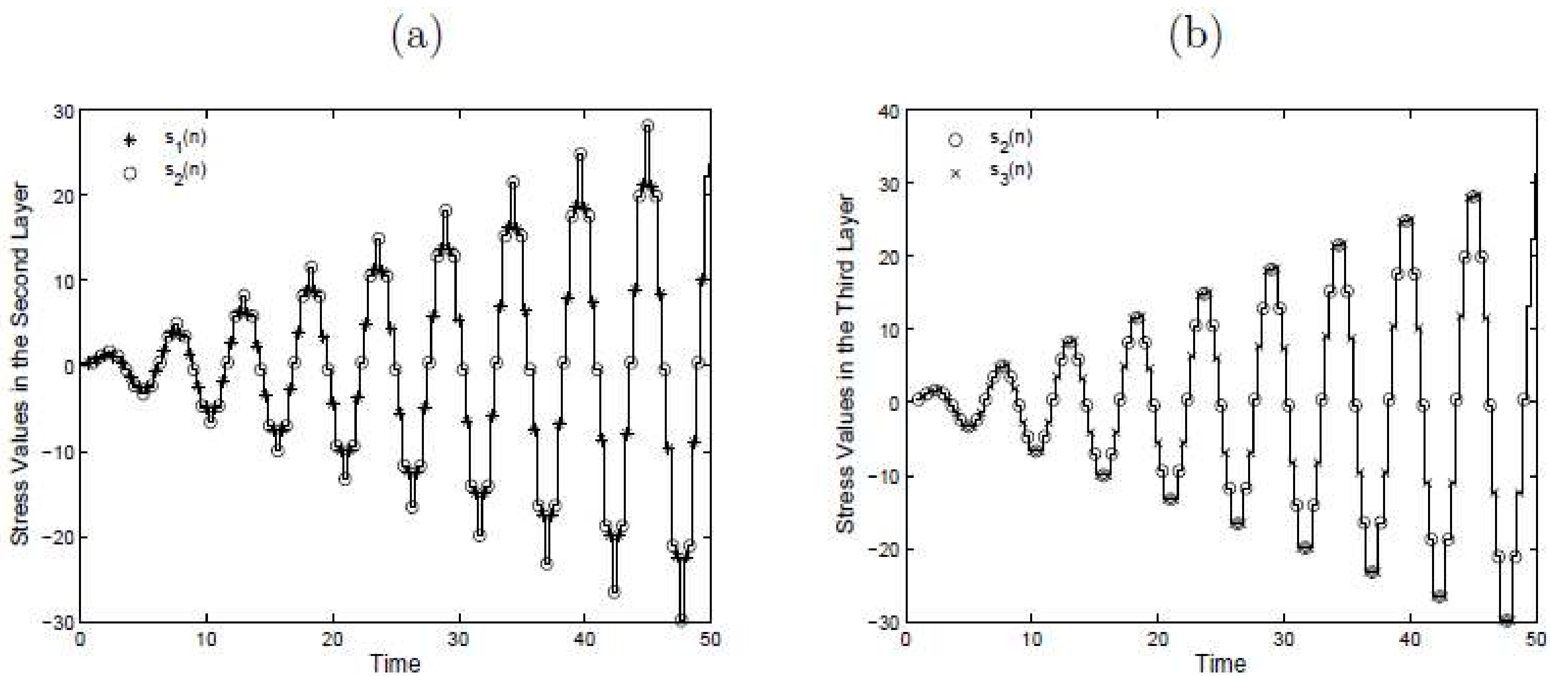
(b) Eleven-layered strip

Stress time history at the middle of the second layer of a Goupillaud-type unit strip subjected to loading $f(n) = \sin(n\pi)$

Values of impedance ratios between two consecutive layers, up to eleven layers:

3, 2, 1.5, 2.2, 0.3, 1.7, 1.4, 3.1, 0.8, 4

Numerical Experiments validating the Resonance Frequencies



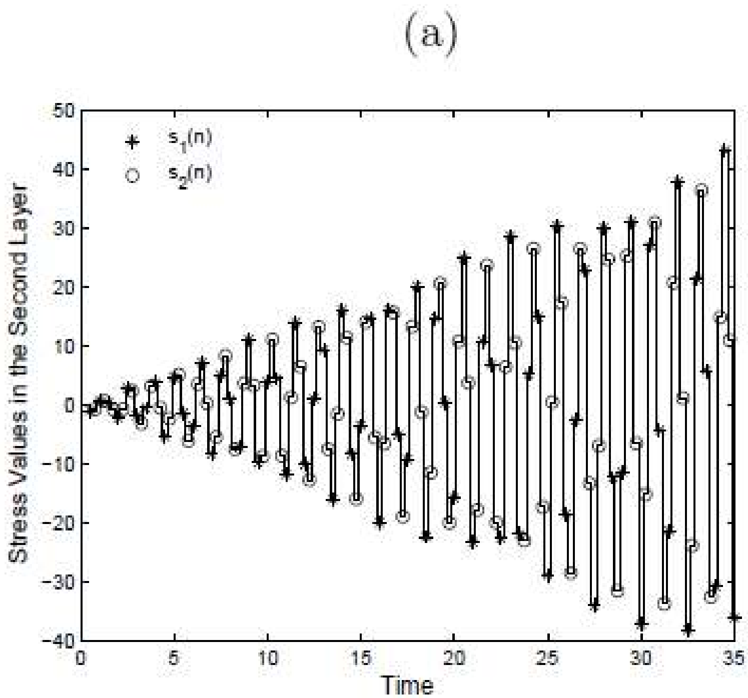
(a) Middle of the second layer (b) Middle of the third layer

Stress time history for a three-layered Goupillaud-type unit strip subjected to loading $f(n) = sin(n\tilde{\omega})$ with $\tilde{\omega} = \theta_1 = \frac{\pi}{4}$

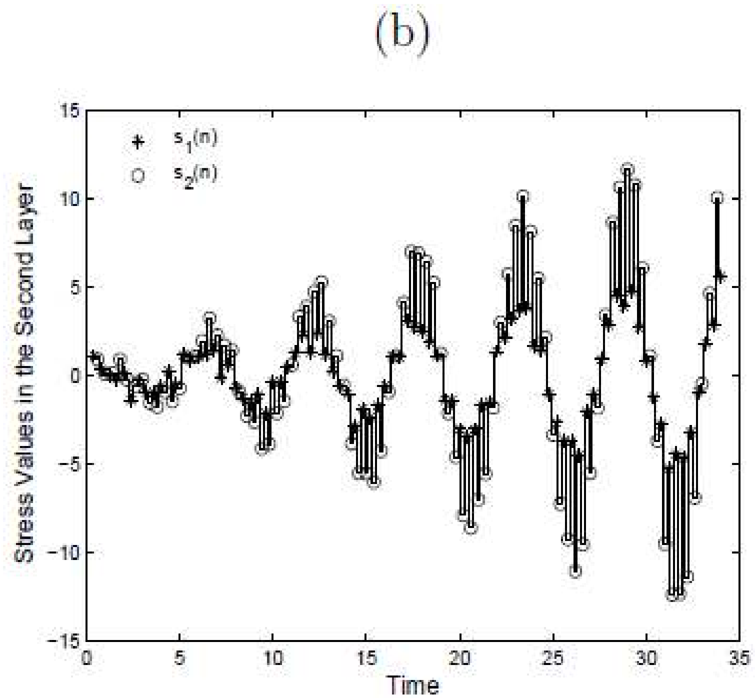
Values of impedance ratios between two consecutive layers up to three layers:

$$\alpha_1 = 3 \qquad \alpha_2 = \frac{1}{2(1-\cos(\pi/4))} - 1$$

Numerical Experiments validating the Resonance Frequencies



(a) Four-layered strip



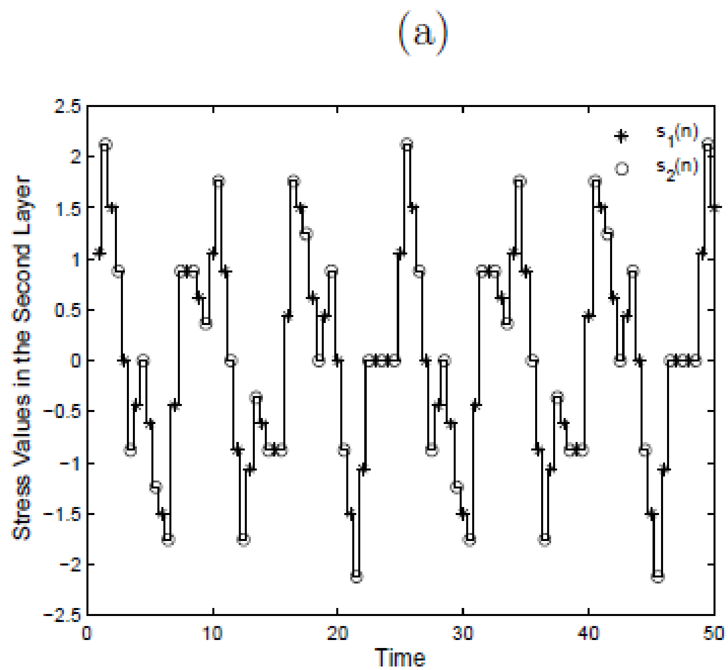
(b) Five-layered strip

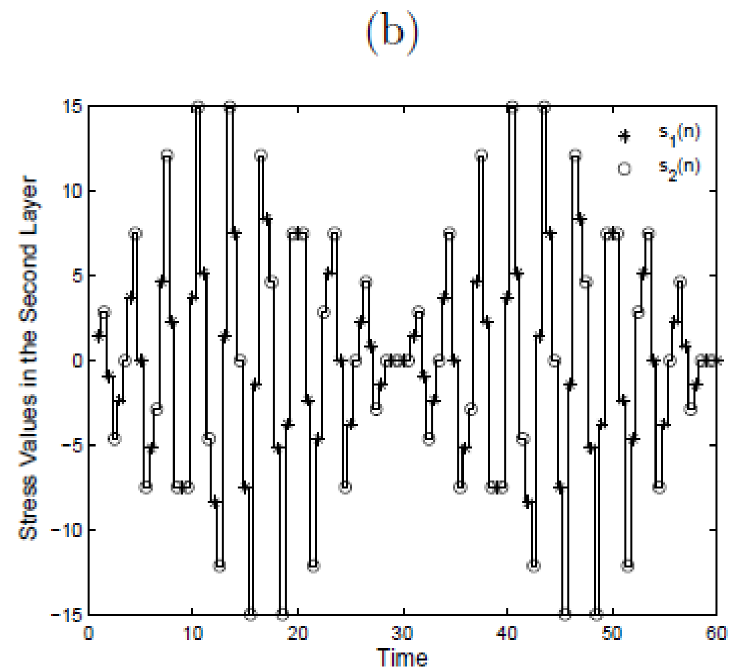
Stress time history at the middle of the second layer of a Goupillaud-type unit strip subjected to loading (a) $\tilde{\omega} = \theta_2 \approx 2.4644 \text{ rad}$ (b) $\tilde{\omega} = \theta_1 \approx 0.453 \text{ rad}$

Values of impedance ratios between two consecutive layers, up to five layers:

0.6, 1.5, 2, 1.2

Numerical Experiments validating the Non-Resonance Frequencies





Stress time history for a two-layered Goupillaud-type unit strip

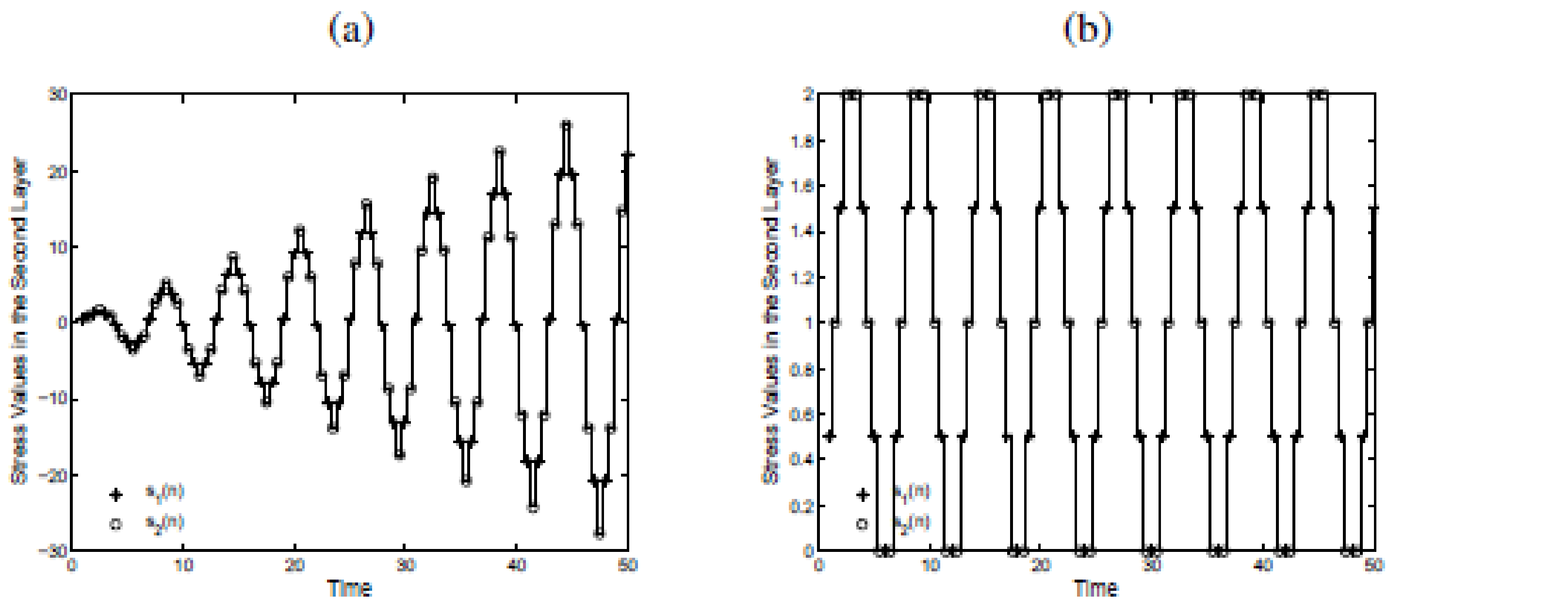
Impedance ratio $\alpha_1 = 1/3$, angle $\theta_1 = 2\pi/3$,

subjected to loading $f(n) = sin(n\tilde{\omega})$ with

(a) $\tilde{\omega} = \pi/4$. (b) $\tilde{\omega} = 0.9 \cdot \theta_1 = 0.6\pi$.

Resonance Frequencies and Optimal Designs

- Optimal Designs that minimize Stress Amplitude under Heaviside Loading, are not immune to the Resonance Phenomena!



Stress time history for a two-layered Goupillaud-type strip with impedance ratio $\alpha_1 = 3$, $\theta_1 = \frac{\pi}{3}$, $\tau = 1$, at the middle of the second layer located at $\xi = 3/4$ subjected to loading (a) $f(n) = \sin(n\tilde{\omega})$ with $\tilde{\omega} = \theta = \frac{\pi}{3}$. (b) $f(n) = 1$ for $n \geq 0$.

Other Applications of our Resonance Frequency Results

- *Design modification that gives a desired frequency spectrum*

  Adding a new layer in front of an existing two-layered Goupillaud-type strip, allows us to modify the resonance frequency spectrum for a three-layered strip.

- *Natural frequencies of a free-fixed non-Goupillaud-type layered strip with integer layer length ratios*

  For instance, the frequency results for a free-fixed four-layered Goupillaud-type strip with equal layer lengths, can be extended to a free-fixed three-layered non-Goupillaud-type strip with wave travel time ratio $1{:}2{:}1$ by choosing $\alpha_2 = 1$.

Summary

- Derived explicit stress solutions from a global system of recursive relationships using *z*-transform methods. The determinant of the system matrix was found to be a palindromic polynomial with real coefficients.

- Optimal Designs

  - Found (theoretically) infinitely many optimal designs.

  - Found designs which offer a time delay benefit to reach the max stress compared to the homogeneous.

  - Made a non-optimal design optimal i.e. made the aluminum-steel two-layered strip an optimal three-layered strip, by adding one more layer of glass fiber reinforced material.

  - The optimization software DYNA3D/GLO corroborated the analytical optimality results by finding the first eight optimal design points to a reasonable degree of accuracy.

- Resonance

  - Represented the resonance frequency spectrum and discussed its properties

- Verified the theoretical predictions with numerical experiments and discussed several applications.

Future Work

- Analytically study the stress wave propagation and vibrations in piezoelectric layered media subjected to a Heaviside voltage.

- Investigate whether all the zeros of the determinant of the system matrix lie on the unit circle for $m>5$ and do the stress optimization for such designs.

References of the Work Presented

- Gazonas, G.A., and Velo, A.P., *z-Transform Methods for the Resonance Frequencies of a Finite Layered Elastic Strip*, in review.

- Velo, A.P., Gazonas, G.A., and Ameya T., *z-Transform Methods for the Optimal Design of One-Dimensional Layered Elastic Media*, SIAM Journal on Applied Mathematics, Vol. 70, No.3, (2009), pg. 762-788.

- Velo, A.P. and Gazonas, G.A., *Stress wave propagation and optimal design of a two layered elastic strip subjected to transient loading*, International Journal of Solids and Structures 40, (2003), pg. 6417-6428.

Acknowledgments

- This research was partially sponsored by the Short Term Analysis Services (STAS) Program, administered by the Battelle Memorial Institute in North Carolina for the U.S. Army Research Laboratory.

- Student collaborators:
  - Takanobu Ameya, International student from Japan, USD
  - Thomas Debaecker, The Military Academy of Saint-Cyr, France
  - Thomas Davis, USD


完