Nonlinear dynamics of two-component Bose-Einstein condensates (BECs)

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Collaborators/Links (I)

- Nonlinear Dynamical Systems @ SDSU: http://nlds.sdsu.edu/
  - Peter Blomgren (Numerical PDEs, image processing)
  - Ricardo Carretero (App. math., nonlinear lattices and waves)
  - Joe Mahaffy (Mathematical biology, delay differential equations)
  - Antonio Palacios (Applied mathematics, bifurcations, symmetries)

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- Research Students involved in BECs/nonlinear waves
  - Ron Caplan, Rafael Navarro, Eunsil Baik (PhD, Comp. Sci.).
  - Carlos Prieto, (MS+PhD, Dyn. Syst.).
  - Recent departures: Manjun Ma (2008, Postdoc), Max Rietmann (2009), Suchitra Jagdish (2009), John Everts (2008), Mike Davis (2007), Chris Chong (2006) (MS, Dyn. Syst.).

Collaborators in Nonlinear Waves/Lattices, BECs (II)

- Solitons, Vortices and Vortex Lattices
  - Panos Kevrekidis (UMass)
  - D. Frantzeskakis (Athens)
  - Peter Engels (WSU)
  - David Hall (Amherst Coll.)
  - Brian Anderson (UoA)
  - W. Królikowski (CUDOS/ANU)
  - Chiara Daraio (CalTech)
  - Michael Bromley (SDSU)
  - Ricardo Chacón (Badajoz)
  - Boris Malomed (Tel Aviv)
  - Faustino Palmero (Sevilla)
  - Jesus Cuevas (Sevilla)
  - Mason Porter (Oxford)
  - Todd Kapitula (UNM)
  - Keith Promislow (SFU/MSU).
  - Lincoln Carr (Col. Sch. Mines)
  - Augusto Rodrigues (Porto)
  - Vladimir Konotop (Lisboa)
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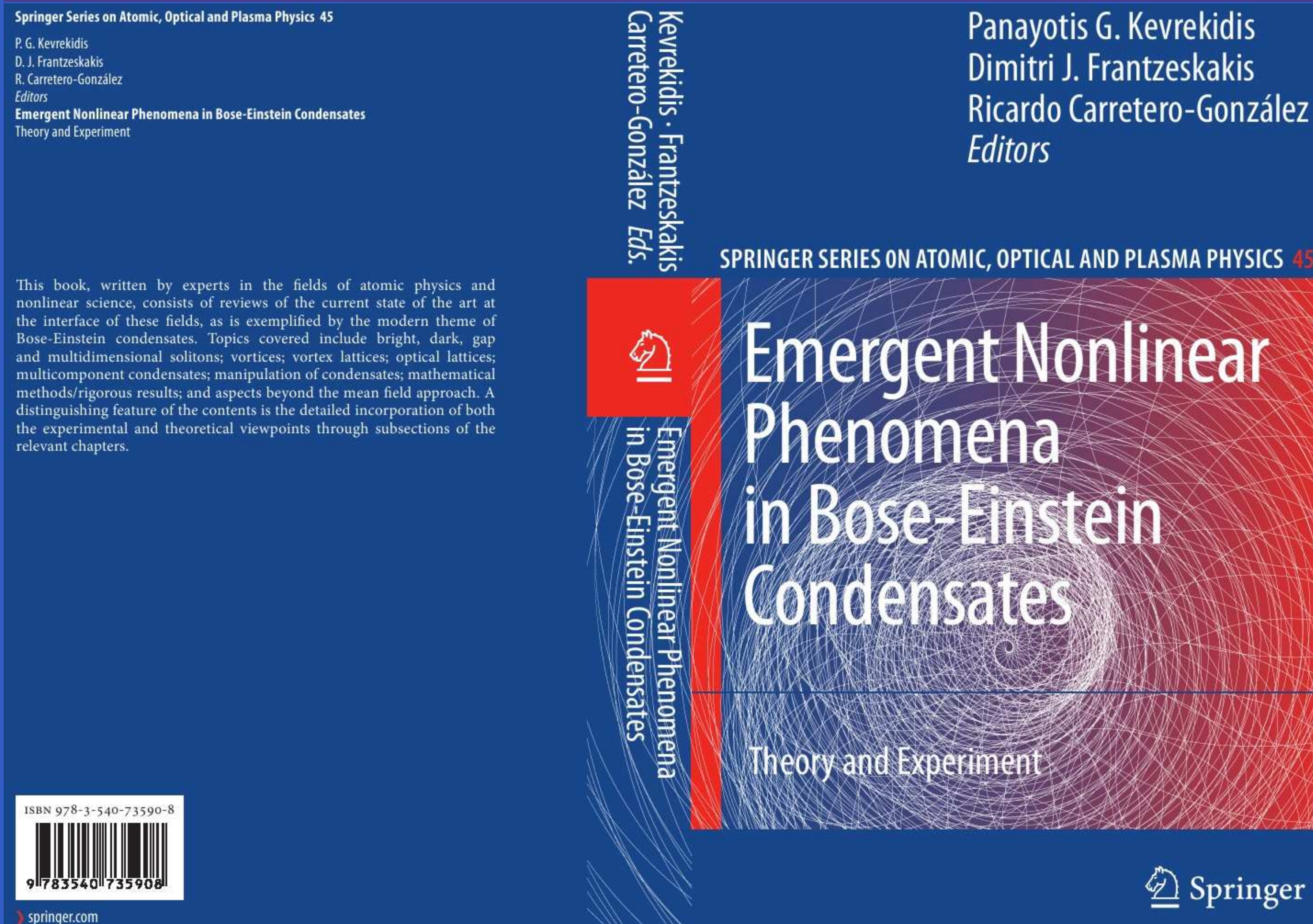
  - Bernard Deconinck (UoW)
  - Enam Hoq (WNEC)
  - Nathan Kutz (UoW)
  - Jared Bronski (UI-UC)
  - Yannis Kevrekidis (Princeton)
  - Dimitri Maroudas (UMass)
  - George Theocharis (UMass)
  - Hector Nistazakis (Athens)
  - Alan Bishop (LANL)
  - Hadi Susanto (Nottingham)
  - Yaroslav Kartashov (ICFO)
  - Lluis Torner (ICFO)
  - etc, ...

New Book — BECs: Theory and Experiment.



Road map

- Introduction
  - Physics of BECs
  - External trapping $\rightarrow$ controlling the dimensionality

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- Nonlinear dynamics in binary BECs
  - Dart-board oscillating patterns in repulsive BECs
  - Symbiotic oscillations of dark-bright solitons
  - Vortex-vortex interactions

Introduction to BECs

Bose-Einstein condensates (BEC)

1925 Bose & Einstein predicted that a gas at very low temperature undergoes quantum “freezing”.

- $T \downarrow \Rightarrow$ vel. $\downarrow \Rightarrow$ de Broglie: $\lambda = h/p \Rightarrow \lambda \uparrow \Rightarrow$ coherence $\Rightarrow$ all atoms enter the SAME quantum state
- BEC is to matter what laser is to light (coherent)
- 5th state of matter (gas + liquid + solid + plasma + BEC)

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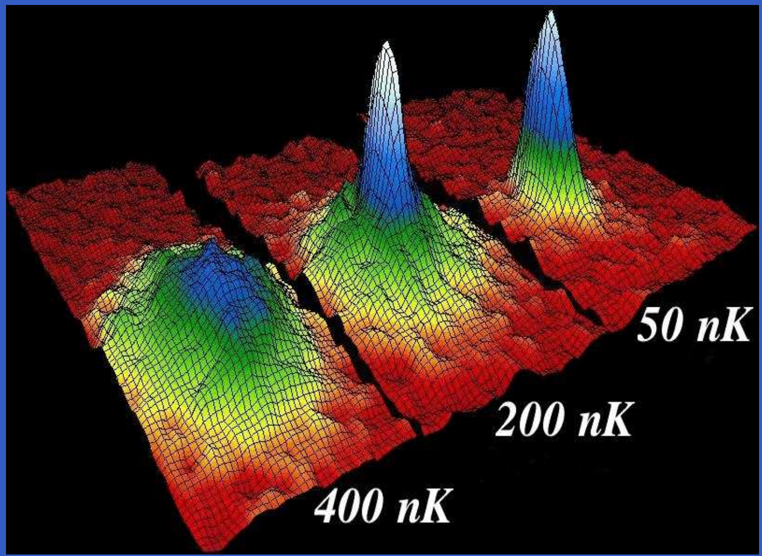
1995 Cornell + Wieman + others @ JILA + NIST + UC) achieved temperatures < 1/170$\overline{M}$ ^{o}K to produce a BEC (rubidium) for the 1st time.



movie (gif), movie (mov)

2001 Cornell + Ketterle + Wieman got the Nobel Prize in Physics for BECs.

2003 Abrikosov + Ginzburg + Leggett got the Nobel Prize in Physics for superconductors and superfluids.

2009 BEC count: Some 60 different BEC experiments.

BECs @ $T = 0$: Many-body Hamiltonian $\rightarrow$ GPE:

- Many-body Hamiltonian for interacting bosons confined in $V_{\rm ext}(\mathbf{r})$

$$\hat{H} = \int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\rm ext}(\mathbf{r}) \right] \hat{\Psi}(\mathbf{r}) + \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}^\dagger(\mathbf{r}') V_{\rm 2B}(\mathbf{r}-\mathbf{r}') \hat{\Psi}(\mathbf{r}') \hat{\Psi}(\mathbf{r}),$$

  - $V_{\rm ext}(\mathbf{r})$: external potential
  - $\hat{\Psi}(\mathbf{r})$ and $\hat{\Psi}^\dagger(\mathbf{r})$ annihilation and creation operators
  - m : mass of bosons
  - $V_{\rm 2B}(\mathbf{r} - \mathbf{r}')$: two-body interatomic potential.

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- Heisenberg equation $i\hbar(\partial\hat{\Psi}/\partial t) = [\hat{\Psi}, \hat{H}]$ gives the dynamics:

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\mathbf{r}) + \int d\mathbf{r}'\hat{\Psi}^\dagger(\mathbf{r}',t)V_{\rm 2B}(\mathbf{r}'-\mathbf{r})\hat{\Psi}(\mathbf{r}',t)\right]\hat{\Psi}.$$

- Binary hard-sphere collisions only: $V_{\rm 2B}(\mathbf{r}'-\mathbf{r}) = g\delta(\mathbf{r}'-\mathbf{r}) \rightarrow$ GPE:

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\mathbf{r}) + g|\Psi(\mathbf{r},t)|^2\right]\Psi(\mathbf{r},t).$$

Gross-Pitaevskii Eq.:

- Close to $T = 0$ BEC $\rightarrow$ Gross-Pitaevskii Eq.:

$$i\hbar \frac{\partial \psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{r}) + gN \left|\psi\right|^2 \right] \psi, \quad (1)$$

  - $\psi(x, y, z, t)$: BEC wavefunction (normalized to unity),
  - $|\psi(x, y, z, t)|^2$: atom density,
  - N: number of atoms,
  - nonlinear coeff: $g = 4\pi\hbar^2 a_s/m$,
  - a_s scattering length:
    - a_s>0 : repulsive : (^{23}Na, ^{87}Rb, H, ^{4}He, ^{85}Rb)
      $\rightarrow$ [dark solitons, vortices]
    - a_s<0 : attractive : (^{7}Li, ^{85}Rb)
      $\rightarrow$ [bright solitons, Bose Nova]
  - $V_{\text{ext}}(\mathbf{r}) = V_{\text{ext}}(x, y, z)$: external confining potential

BECs inside magnetic trap: changing the dimension

- 3D $\rightarrow$ 2D: pancake trap external (confining) magnetic potential:

$$V_{\rm ext}(x,y,z) \quad = \quad \frac{1}{2}m\omega_r^2 r^2 + \frac{1}{2}m\omega_z^2 z^2,$$

  - $r^2 = x^2 + y^2$: in-plane dim., z: strong confining dir.
  - $\omega_z \gg \omega_r$: Quasi-2D BEC (pancake shaped) $\Rightarrow$ vortices

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- 3D $\rightarrow$ 1D: cigar-shaped external (confining) magnetic potential:

$$V_{\rm ext}(x,y,z) \quad = \quad \frac{1}{2}m\omega_x^2 x^2 + \frac{1}{2}m\omega_r^2 r^2,$$

  - $r^2 = y^2 + z^2$: transverse dimensions
  - $\omega_x \ll \omega_r$: Quasi-1D BEC (cigar shaped) $\Rightarrow$ solitons

Two-component BECs:

phase separation

Binary BECs

- Full eqs:

$$i\hbar\frac{\partial\psi_1}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_1 + a_{11}\left|\psi_1\right|^2 + a_{12}\left|\psi_2\right|^2\right)\psi_1,$$

$$i\hbar\frac{\partial\psi_2}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_2 + a_{22}\left|\psi_2\right|^2 + a_{21}\left|\psi_1\right|^2\right)\psi_2.$$

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- Non-dim. eqs:

$$i\frac{\partial u_1}{\partial t} = \left(-\frac{1}{2}\nabla^2 + V_1 + g_{11}\left|u_1\right|^2 + g_{12}\left|u_2\right|^2\right)u_1,$$
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- 2 hyperfine states of the same atom (cf. ^{87}Rb in D. Hall's group) $\Rightarrow$ g's and V's are approximately the same ($g_{ij} \approx 1$ in adim eqs.).
- 2 different types of atoms (cf. ^{41}K-^{87}Rb and ^{7}Li-^{133}Cs) $\Rightarrow g_{11} \neq g_{22}$.
- Symmetry: always $g_{12} = g_{21}$.
- With external fields (Feshbach resonance) it is possible to tune g_{ij}'s.

1D (same species) binary BEC in parabolic trap

- Our equation (1D + 2 hyperfine state of *same* atoms, $g \equiv g_{12}/g_{11}$):

$$i\frac{\partial u_1}{\partial t} = \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\Omega^2}{2}x^2 + |u_1|^2 + g\,|u_2|^2\right)u_1,$$

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- Two “forces” competing: BEC repulsion vs. trap attraction (draw it)

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- Two “forces” competing: BEC repulsion vs. trap attraction (draw it)
- Variational approximation (VA):
  - ψ is a solution to NLS $\Leftrightarrow$ ψ extremum of Lagrangian
  - If $\psi = \psi_p$ (ansatz) $\Rightarrow$ NLS reduces to an ODE on the params p through the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L_p}{\partial \dot{p}_i}\right) = \frac{\partial L_p}{\partial p_i}$$

    where $p_i = p_i(t)$ = params describing ansatz's solution.
  - Explain graphically the projection of orbits to a lower-dim manifold.

Method: Variational Approximation (VA)

2C Lagrangian: $\mathcal{L} = \int_{-\infty}^{\infty} (L_1 + L_2 + L_{12} + L_{21})\, dx,$

$$
\begin{aligned}
L_j &= E_j + \frac{i}{2}\left(u_j \frac{\partial\, u_j^*}{\partial t} - u_j^* \frac{\partial u_j}{\partial t}\right), \\
E_j &= \frac{1}{2}\left|\frac{\partial u_j}{\partial x}\right|^2 + V(x)\,|u_j|^2 + \frac{1}{2}\,|u_j|^4\,, \\
L_{12} &= L_{21} = \frac{1}{2} g\,|u_1|^2\,|u_2|^2\,,
\end{aligned}
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Gaussian ansatz:

$$\begin{aligned} u_1(x,t) &= A e^{-\frac{(x-B)^2}{2W^2}} e^{i(C+Dx+Ex^2)}, \\ u_2(x,t) &= A e^{-\frac{(x+B)^2}{2W^2}} e^{i(C-Dx+Ex^2)}. \end{aligned}$$

Time depend. params: $A(t)$, $B(t)$, $C(t)$, $D(t)$, $E(t)$, $W(t)$.

GPE → ODEs

Lagrangian evaluated @ ansatz + Euler-Lagrange eqs:

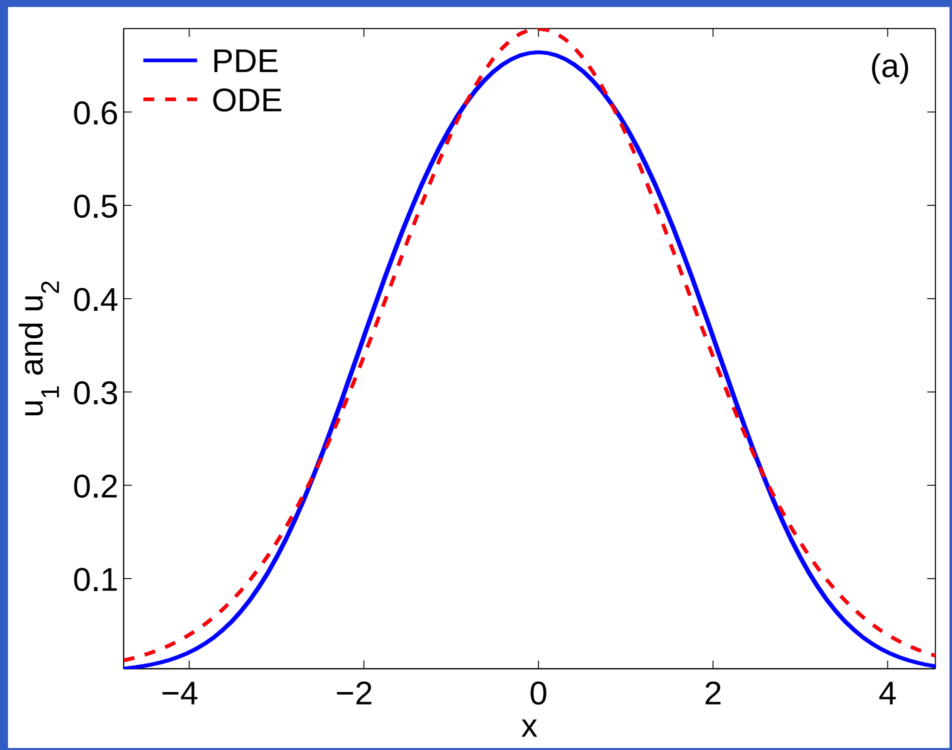
$$
\begin{aligned}
\frac{dA}{dt} &= -AE, \\
\frac{dB}{dt} &= D + 2BE, \\
\frac{dC}{dt} &= \frac{B^2}{2W^2} - \frac{D^2}{2} - \frac{1}{2W^2} + \frac{\sqrt{2}A^2}{8W^2}(2B^2 - 5W^2) + \\
&\quad \frac{\sqrt{2}A^2 g}{8W^2} e^{-\frac{B^2}{2W^2}}(8B^4 + 2B^2W^2 + 5W^4), \\
\frac{dD}{dt} &= \frac{\sqrt{2}A^2 Bg}{2W^4} e^{-\frac{B^2}{2W^2}}(4B^2 + W^2) - \frac{\sqrt{2}A^2 b}{2W^2} - \frac{B}{W^4} - 2DE, \\
\frac{dE}{dt} &= \frac{\sqrt{2}A^2 g}{4W^4} e^{-\frac{B^2}{2W^2}}(-4B^2 + W^2) + \frac{\sqrt{2}A^2}{4W^2} + \frac{1}{2W^4} - 2E^2 - \frac{\Omega^2}{2}, \\
\frac{dW}{dt} &= 2EW.
\end{aligned}
$$

Statics: steady states: mixed vs separated

$$B_* = 0,$$

$$A_*^2 = 2\sqrt{2}\left(8\mu - \sqrt{15\Omega^2 + 4\mu^2}\right)/[15(1+g)],$$

$$W_*^2 = \left(2\mu + \sqrt{15\Omega^2 + 4\mu^2}\right)/[5\Omega^2].$$

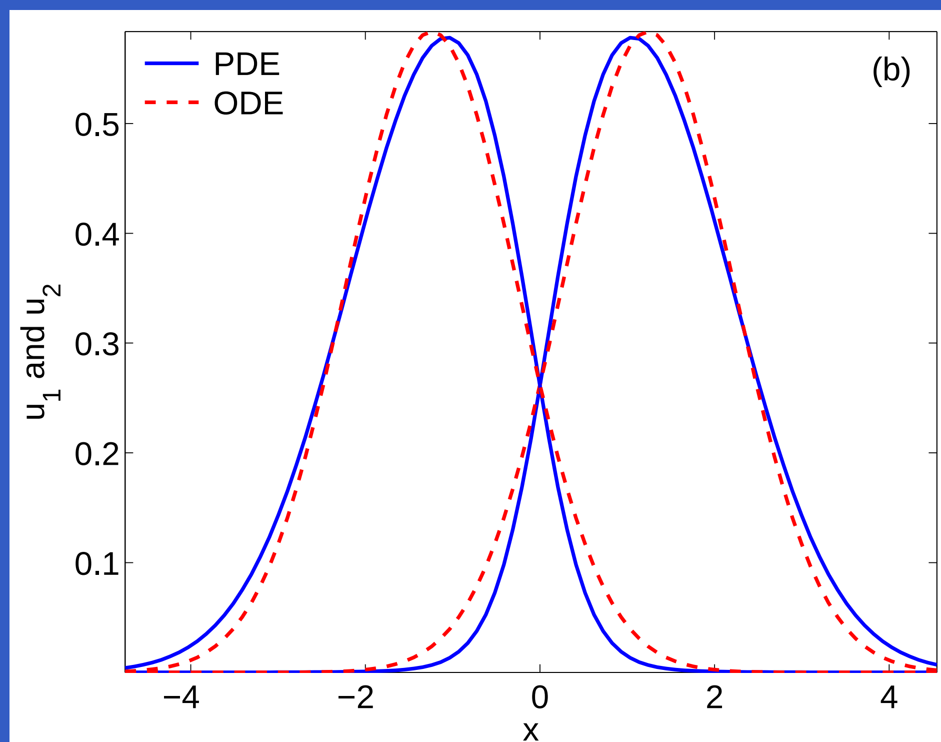


Statics: steady states: mixed vs separated

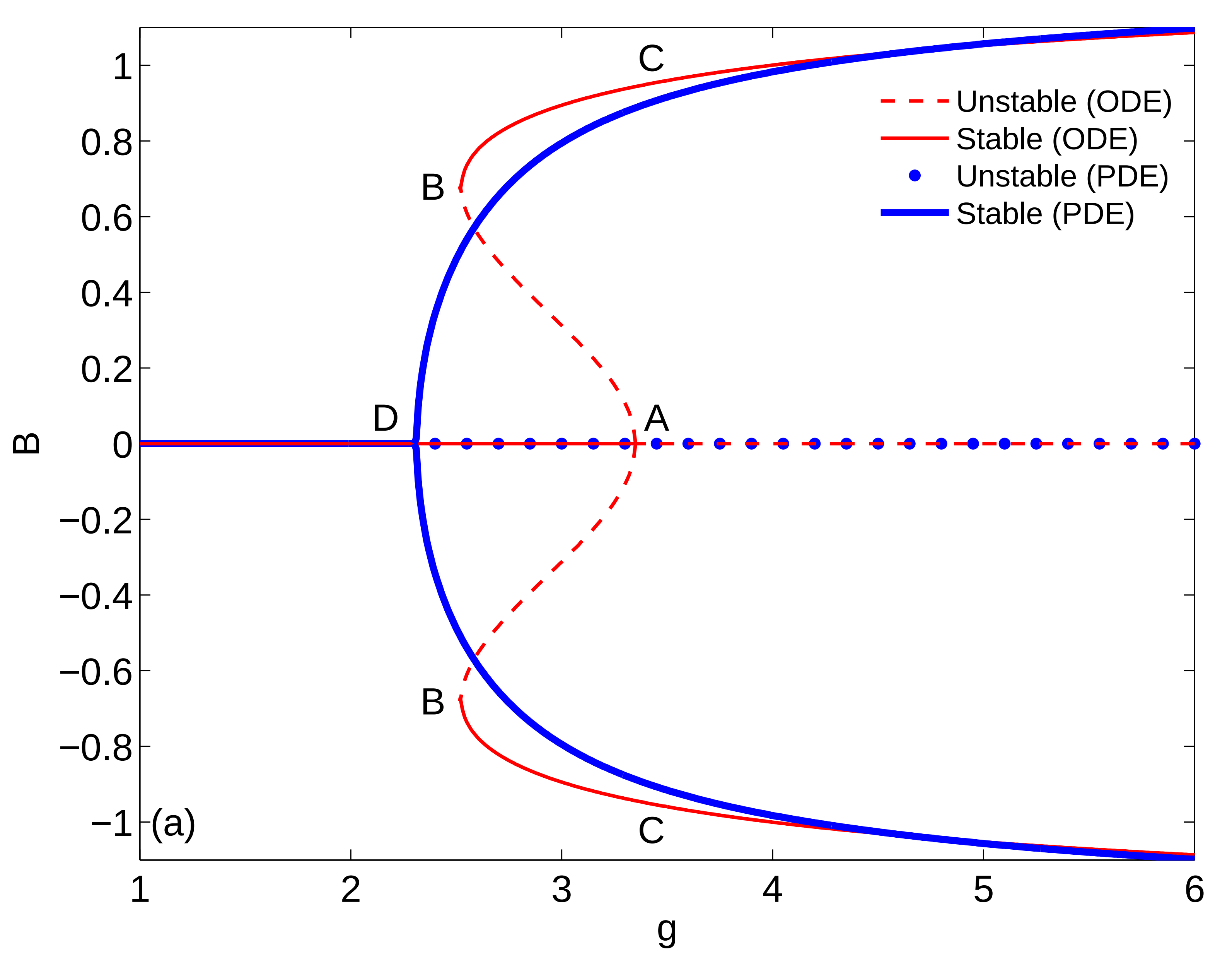
$$\Omega^2 - \frac{\sqrt{2}A_*^2 g}{W_*^2} e^{-\frac{B_*^2}{2W_*^2}} = 0,$$

$$\mu - 1/(2W_*^2) - 5W_*^2\left(\sqrt{2}A_*^2 g + \Omega^2 W_*^2\right)/8 = 0,$$

$$\mu + 3/(4W_*^2) - 5\Omega^2\left(W_*^2 + 2B_*^2\right)/4 = 0.$$

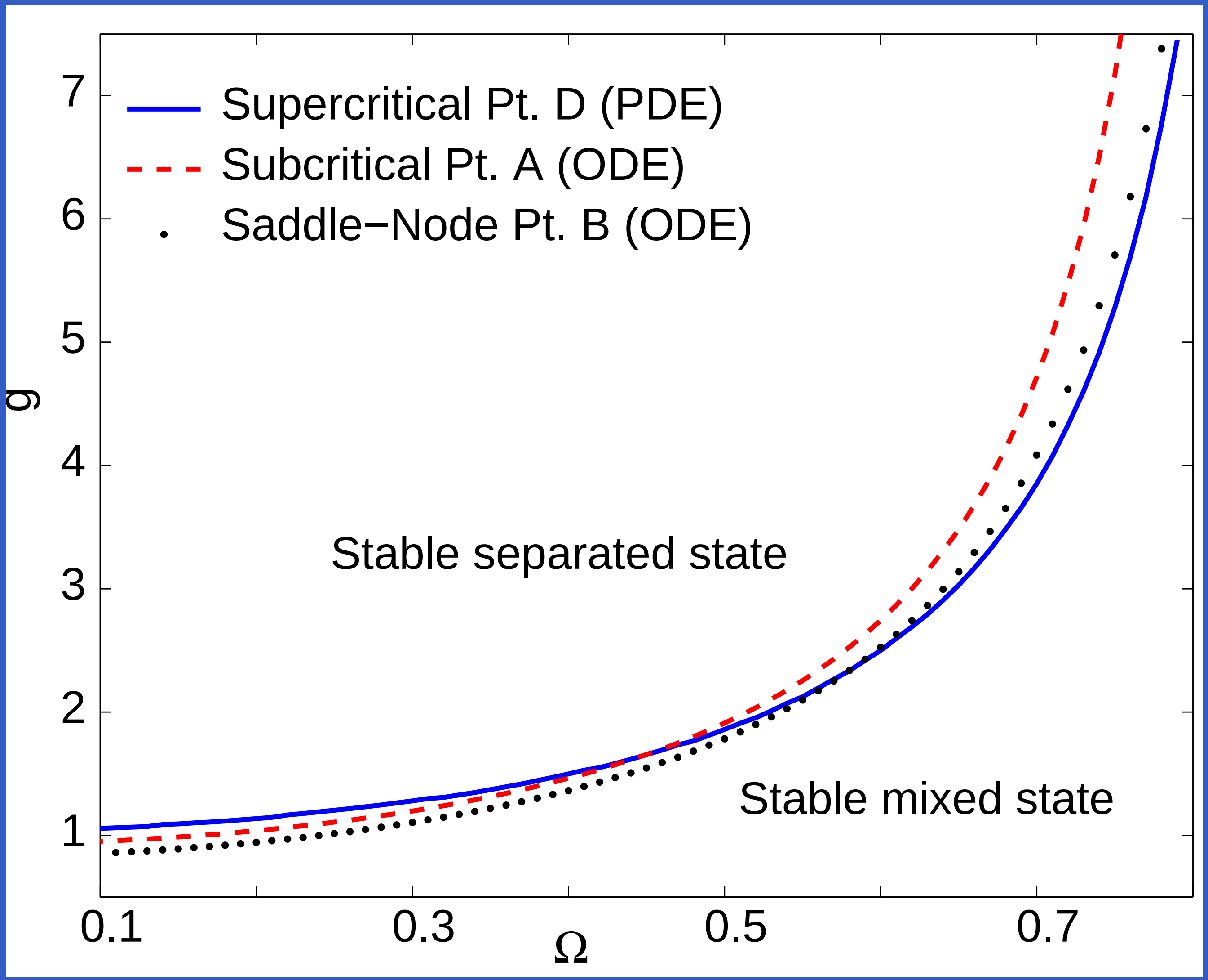


Bifurcation of steady states: PDE vs ODE

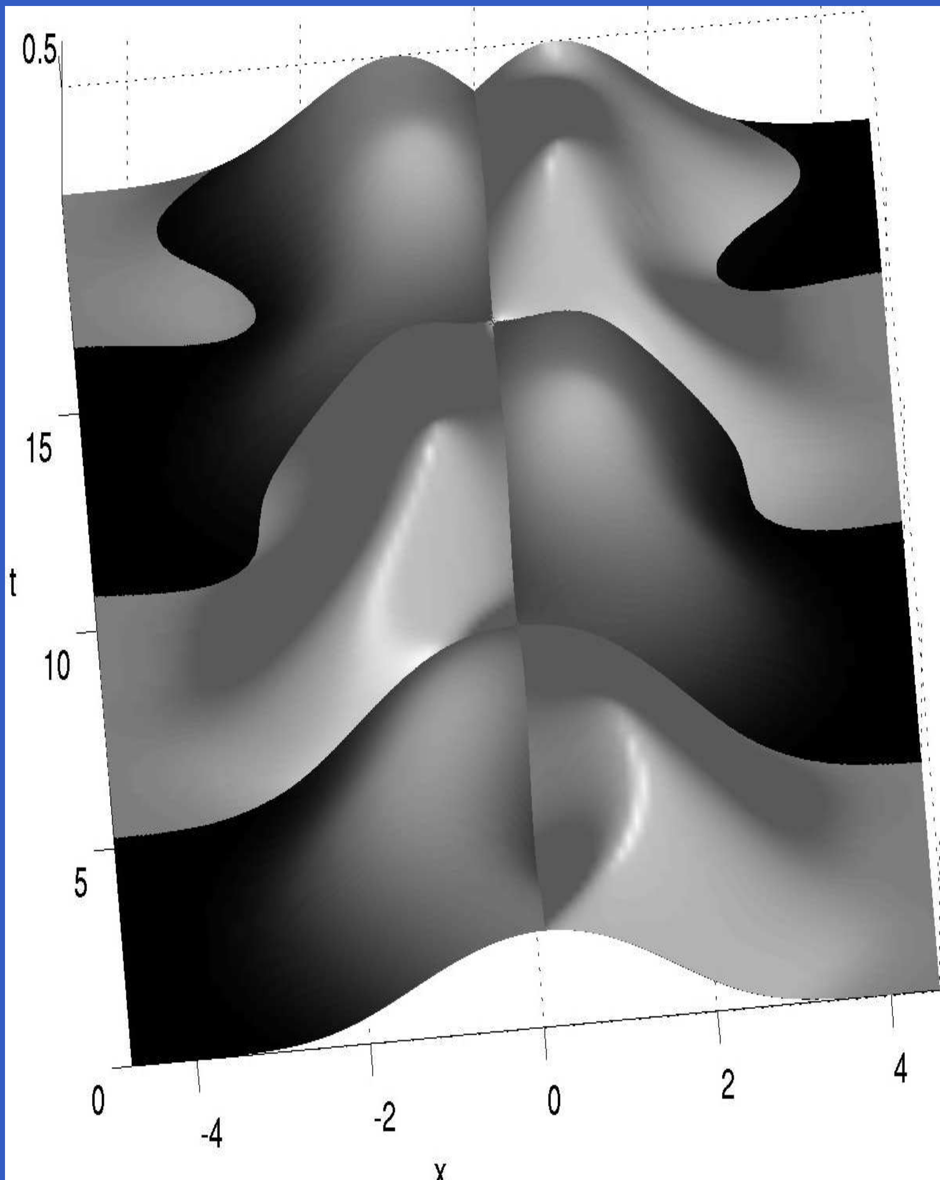


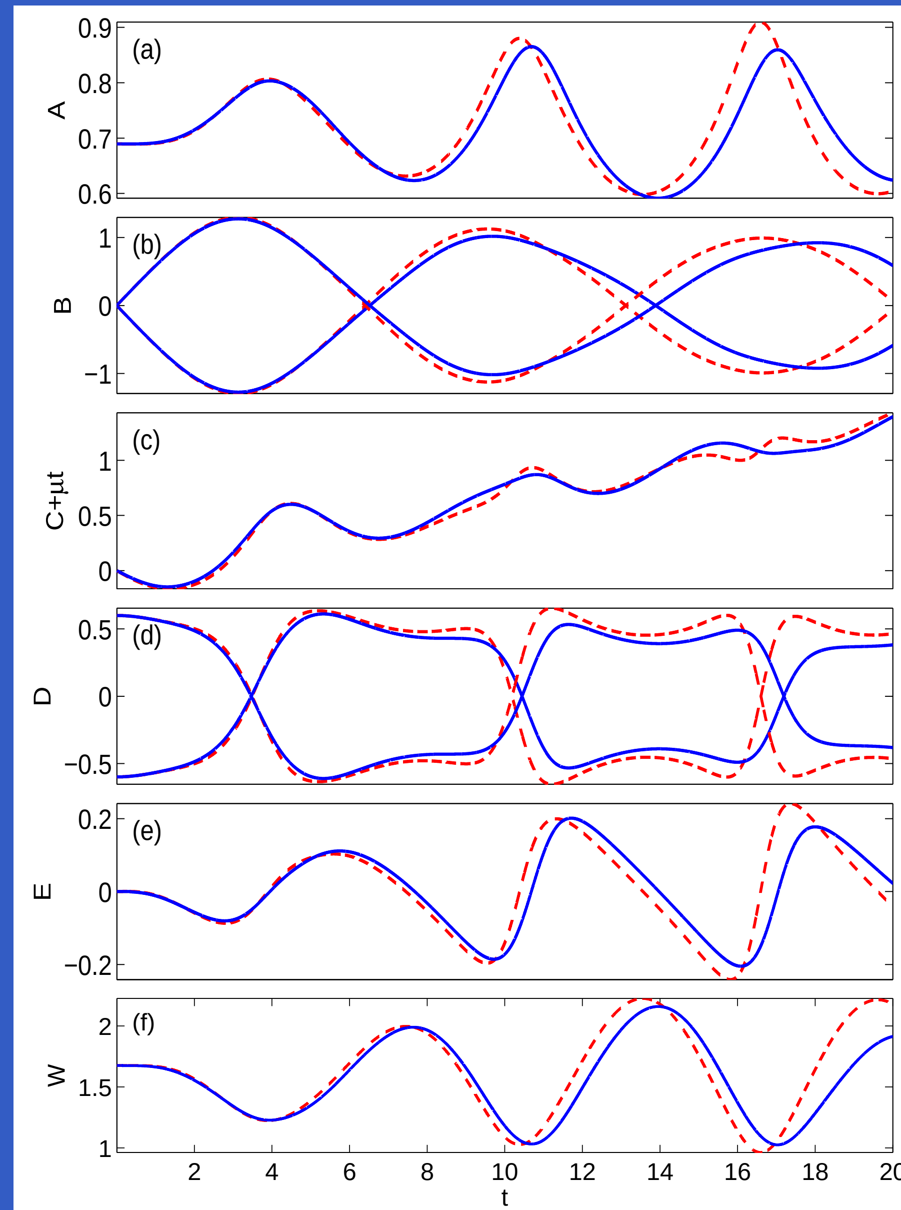
Phase separation

Using pt A : $\quad g_{\text{cr}} = \left(6\mu + 3\sqrt{15\Omega^2 + 4\mu^2}\right) / \left(26\mu - 7\sqrt{15\Omega^2 + 4\mu^2}\right).$

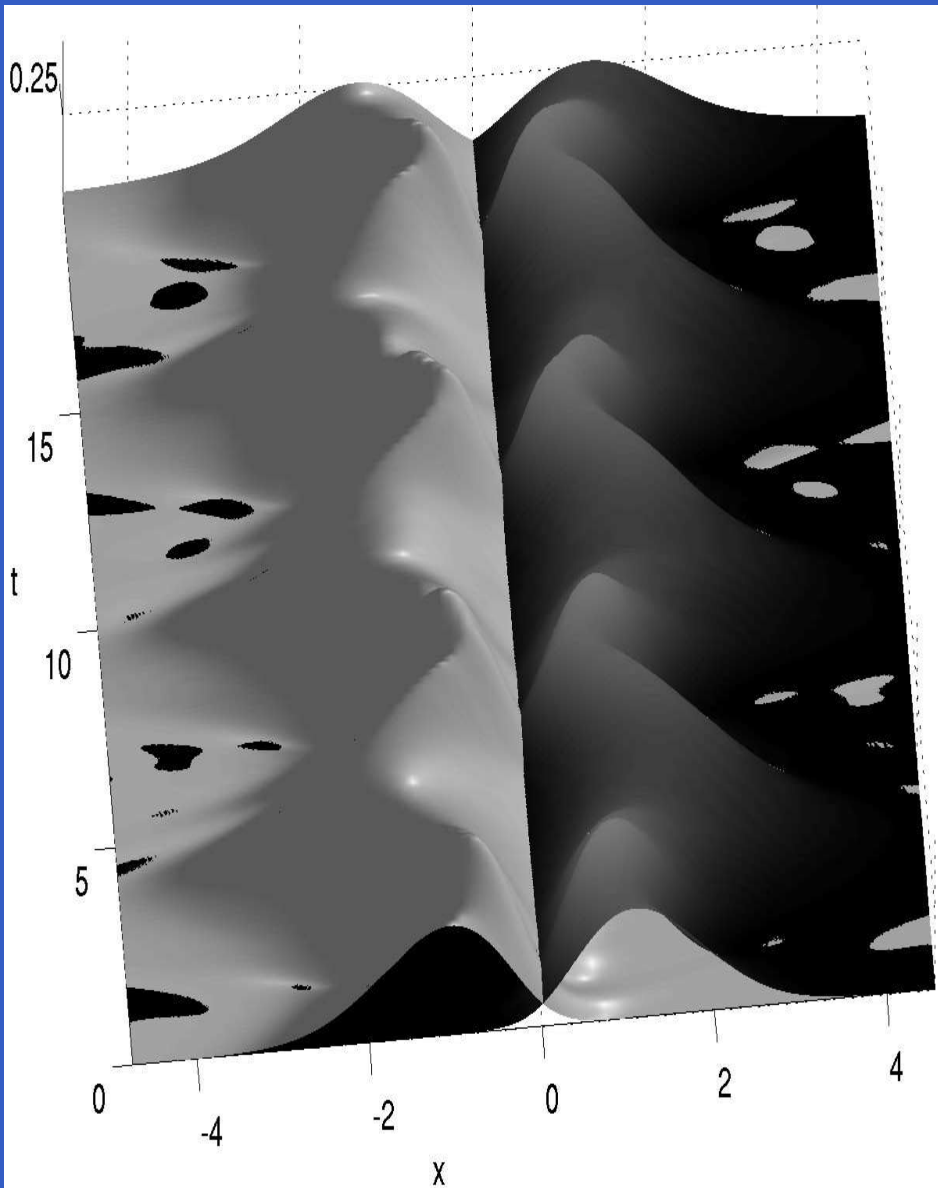


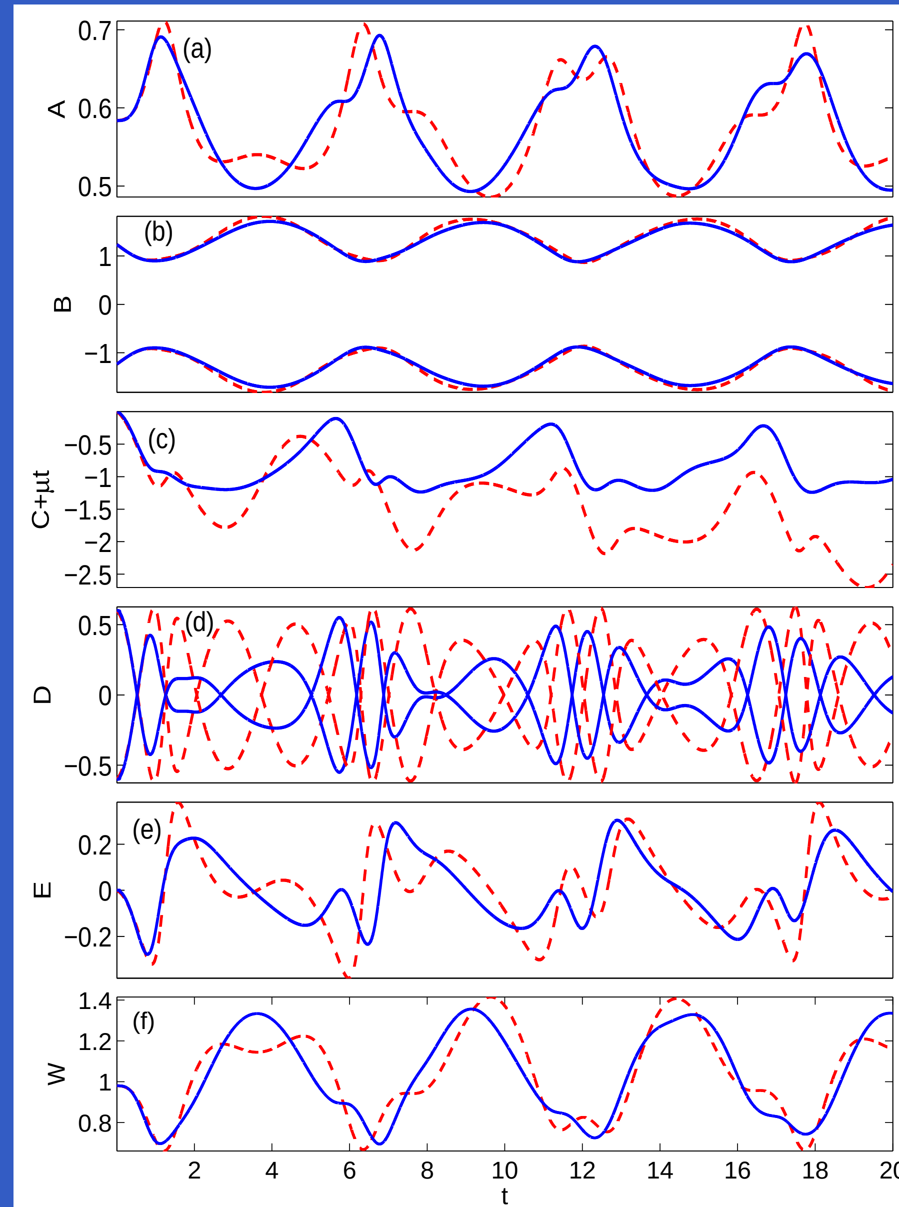
Dynamics: oscillations trough each other





Dynamics: oscillations about fixed pts: separated osc.





Understanding the phase separation $\to U_{\text{eff}}$

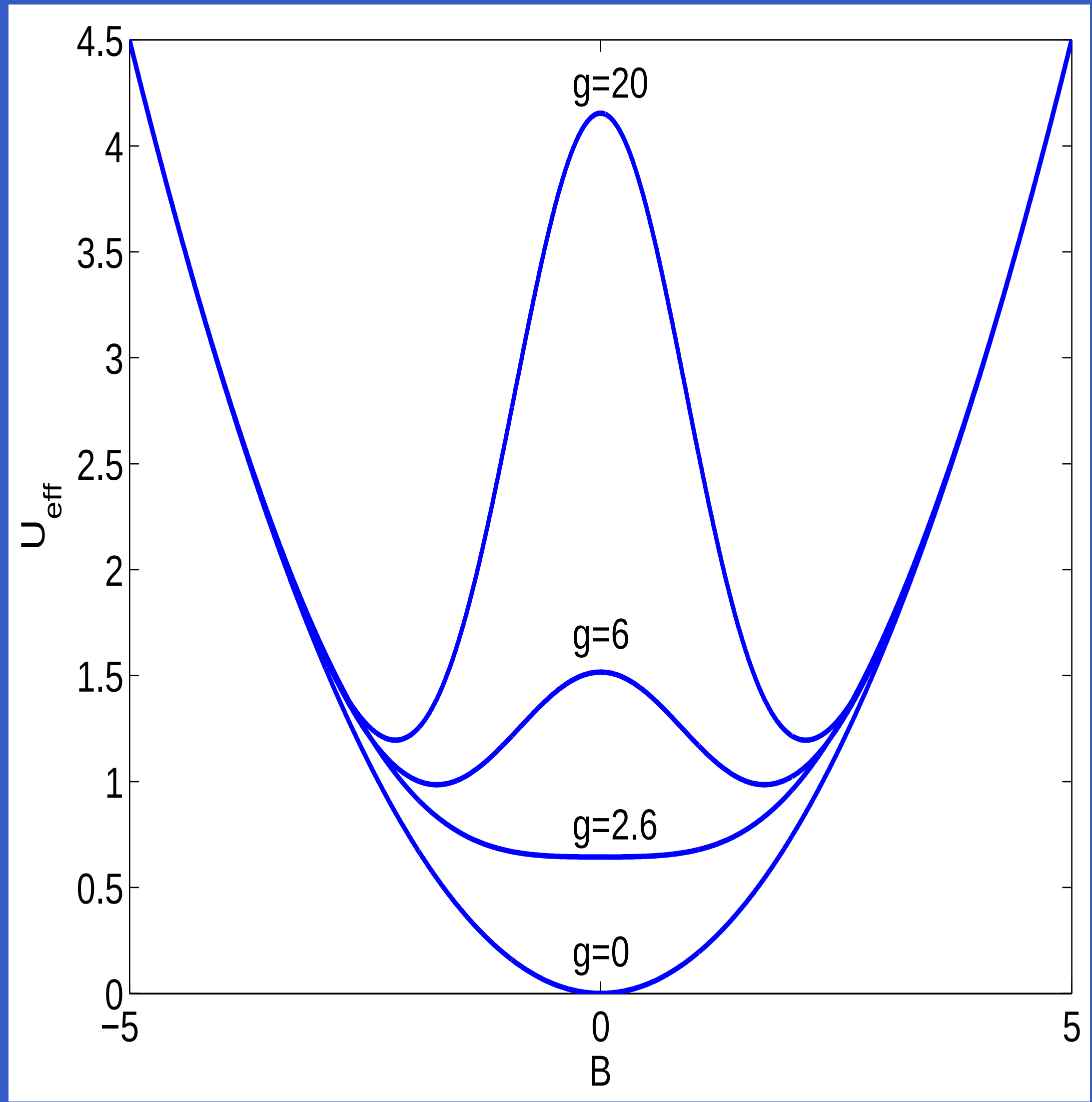
- Newtonian reduction:
  $A(t) \approx A_*$ and $W(t) \approx W_*$
- Newton oscillations

$$\frac{d^2B}{dt^2} = -\frac{dU_{\text{eff}}(B)}{dB},$$

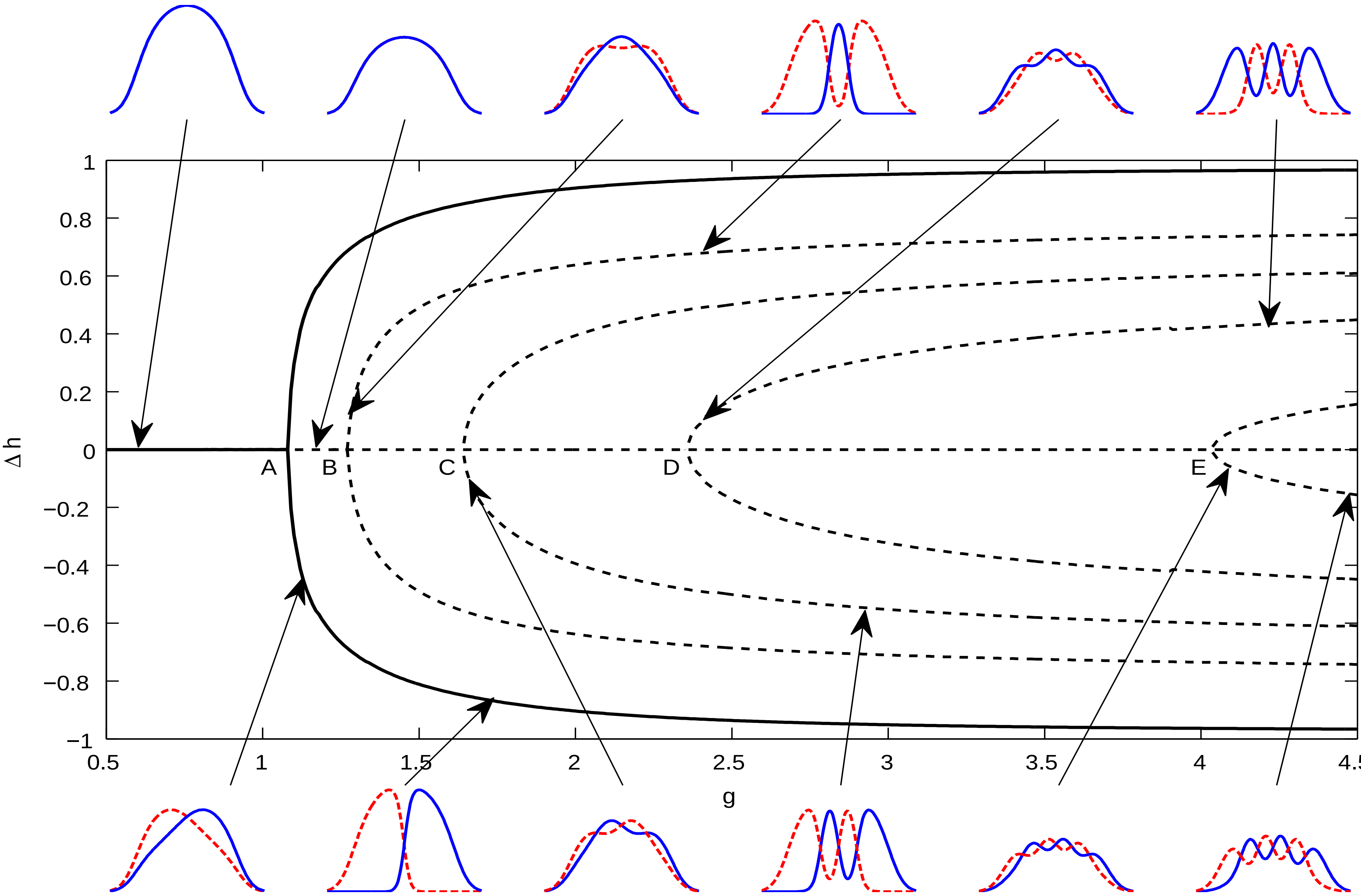
- inside the effective potential

$$U_{\text{eff}} = \frac{\Omega^2}{2}B^2 + \frac{\sqrt{2}A_*^2 g}{2} e^{-\frac{B^2}{2W_*^2}}.$$

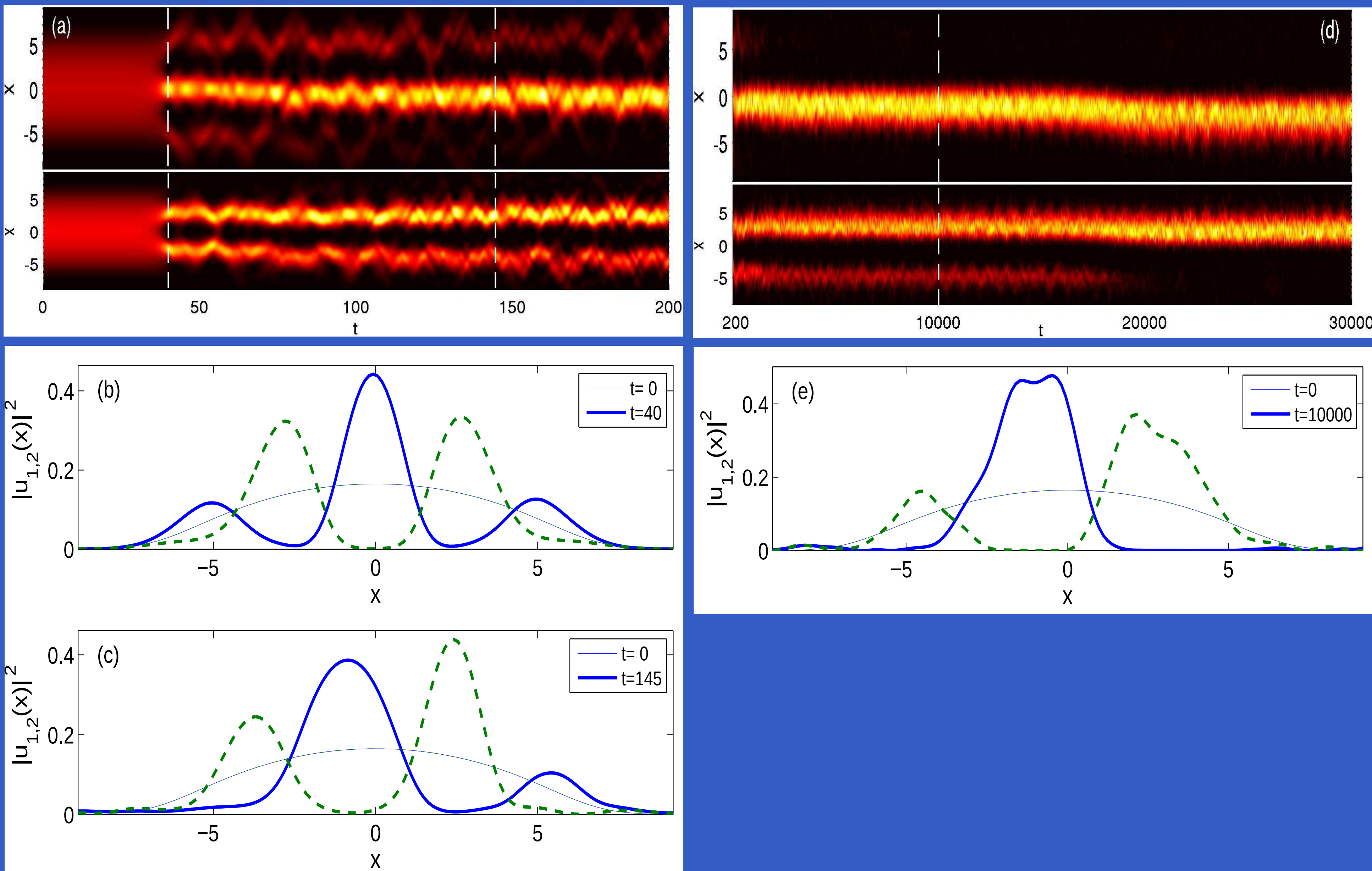
- which becomes double well
  for large enough g ($g > g_{\text{cr}}$)



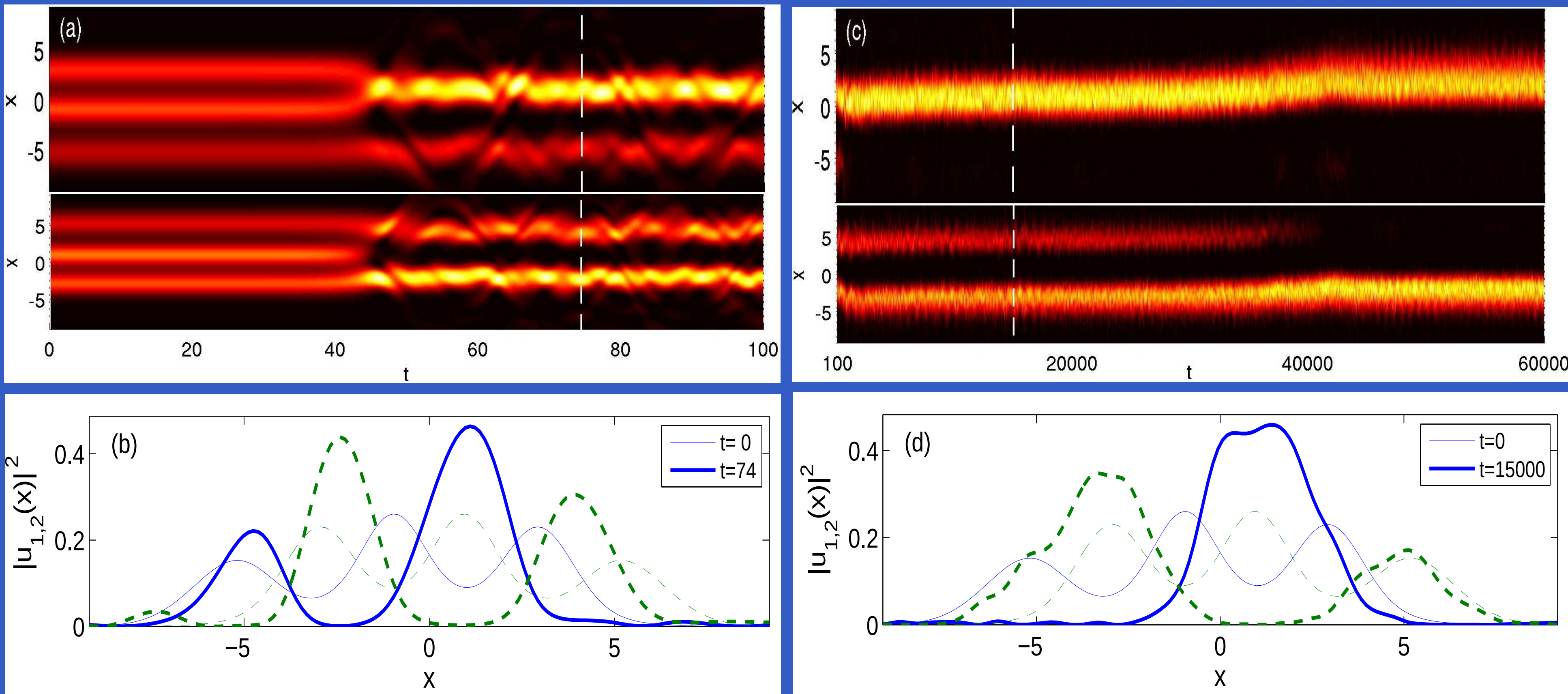
Bifurcation of steady states



Dynamics of unstable mixed state (for $g > g_{\rm cr}$)

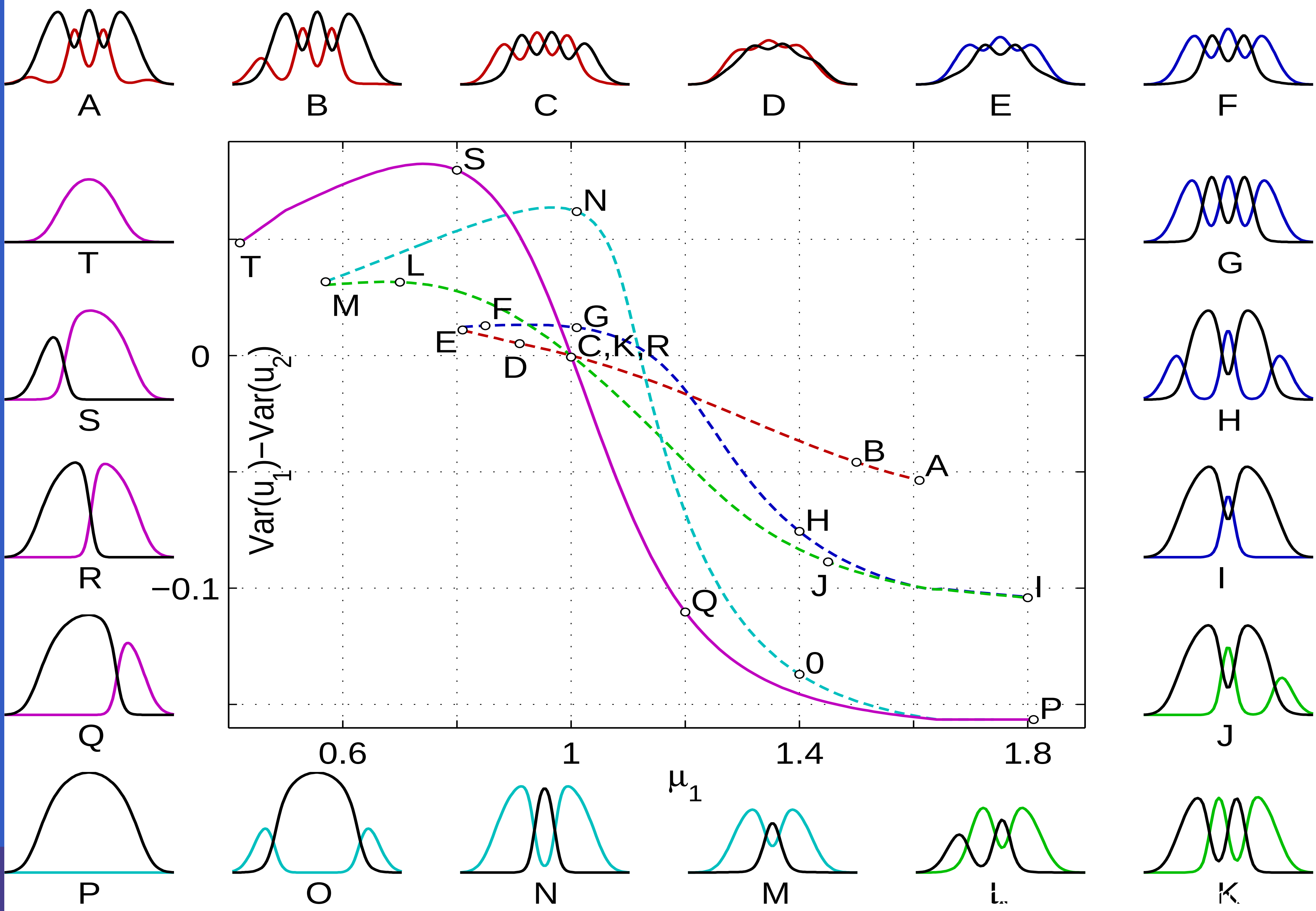


Dynamics of (unstable) higher order states



- Same qualitative ‘decay’ for other states
- All unstable states ‘decay’ to a highly perturbed separated state.

Bifurcation scenario for asymmetric states ($\mu_1 \neq \mu_2$)



Dartboard oscillating patterns

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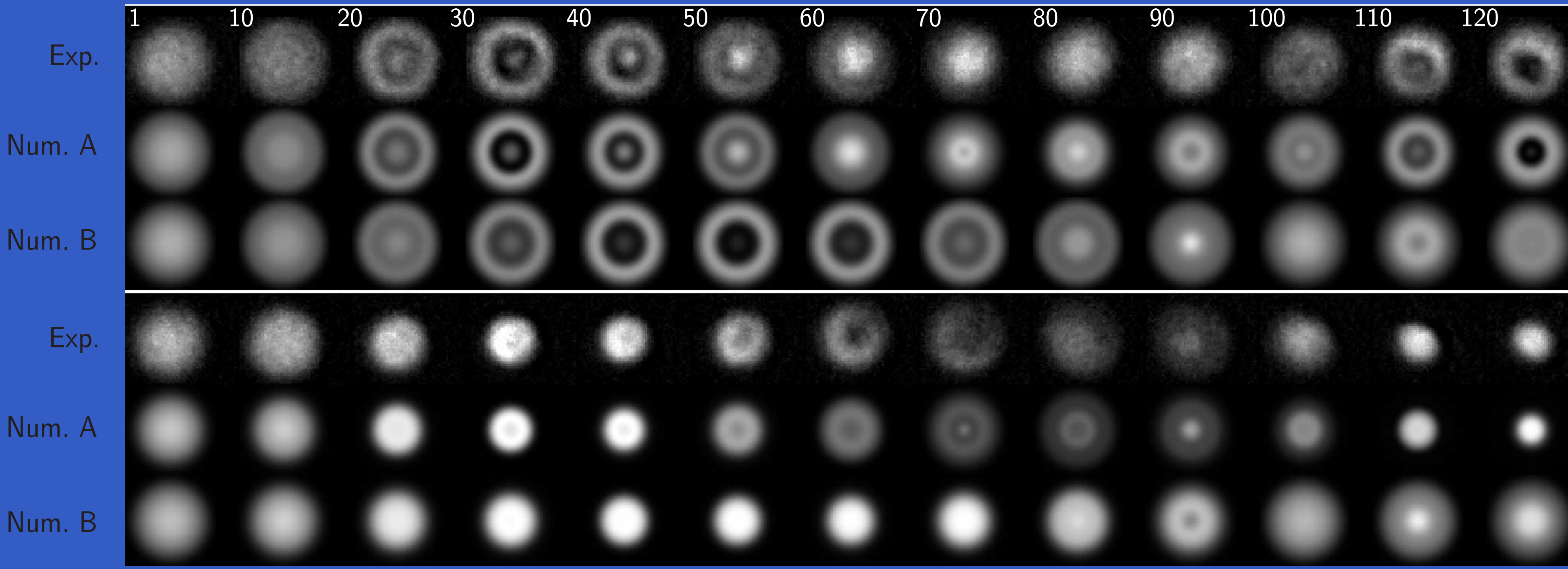
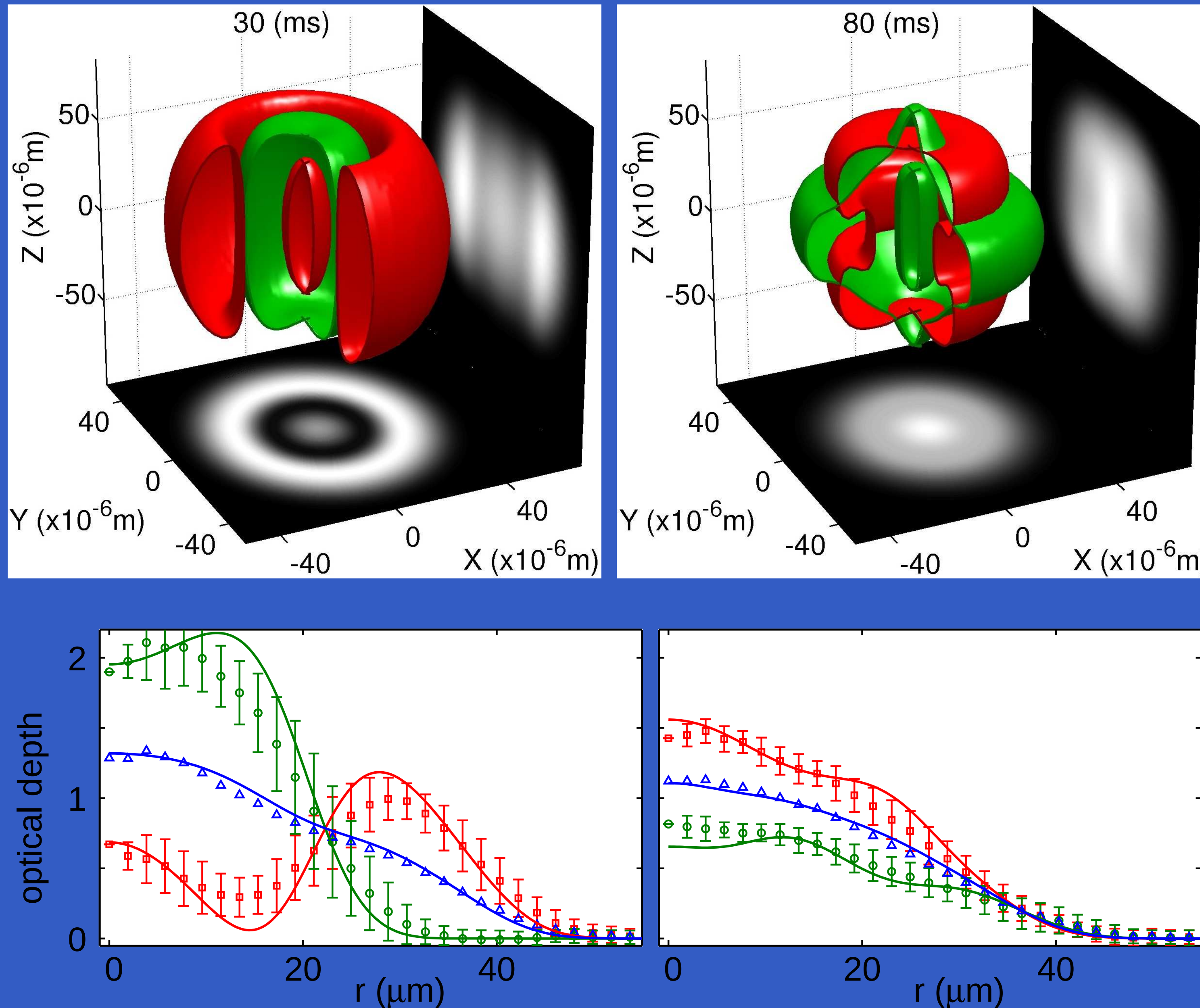
- We are developing a 2D and 3D versions of this work to capture:
  - the ring dartboard patterns.



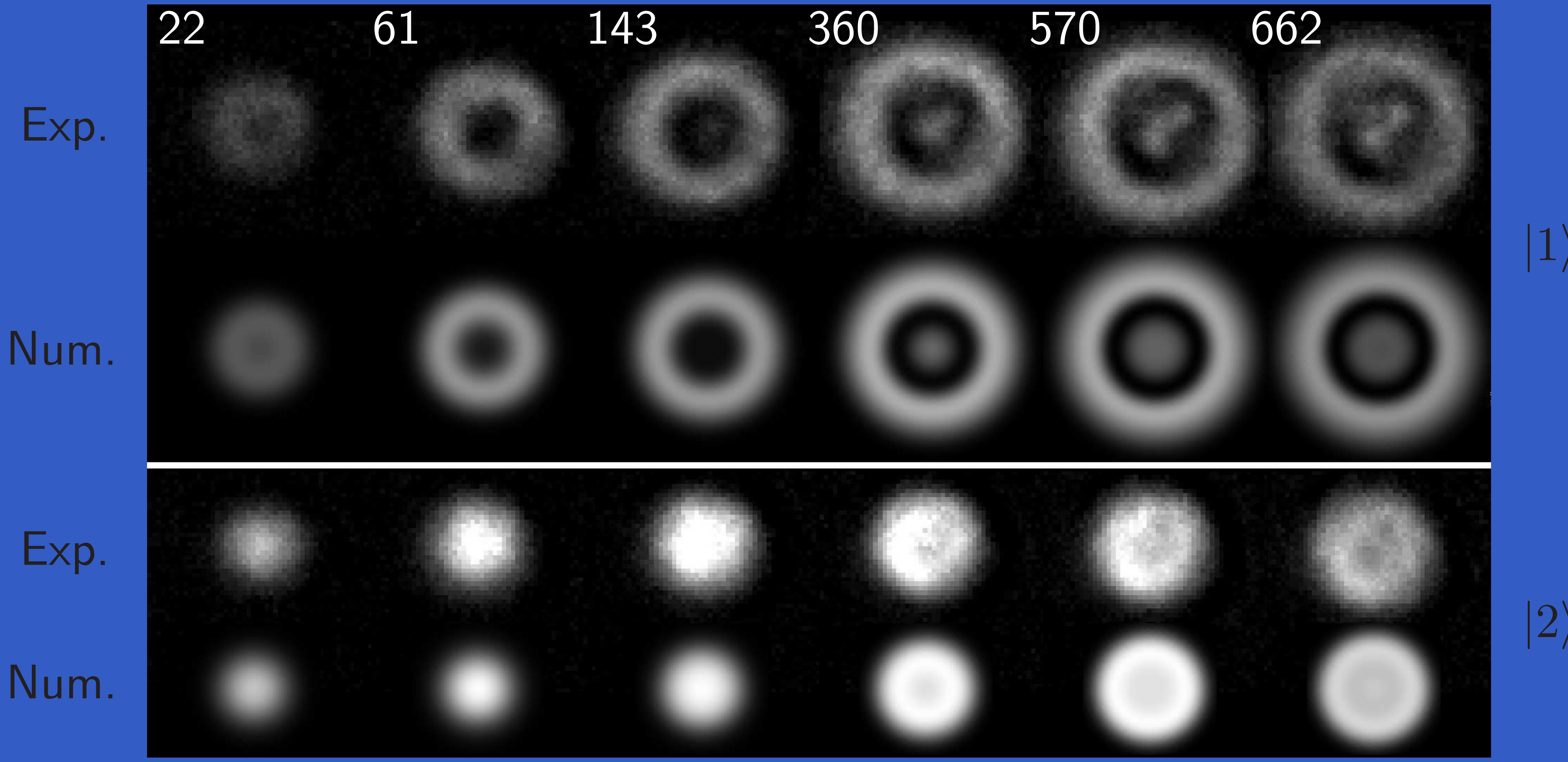
Dartboard oscillating patterns

- Ring oscillation dynamics [movie]



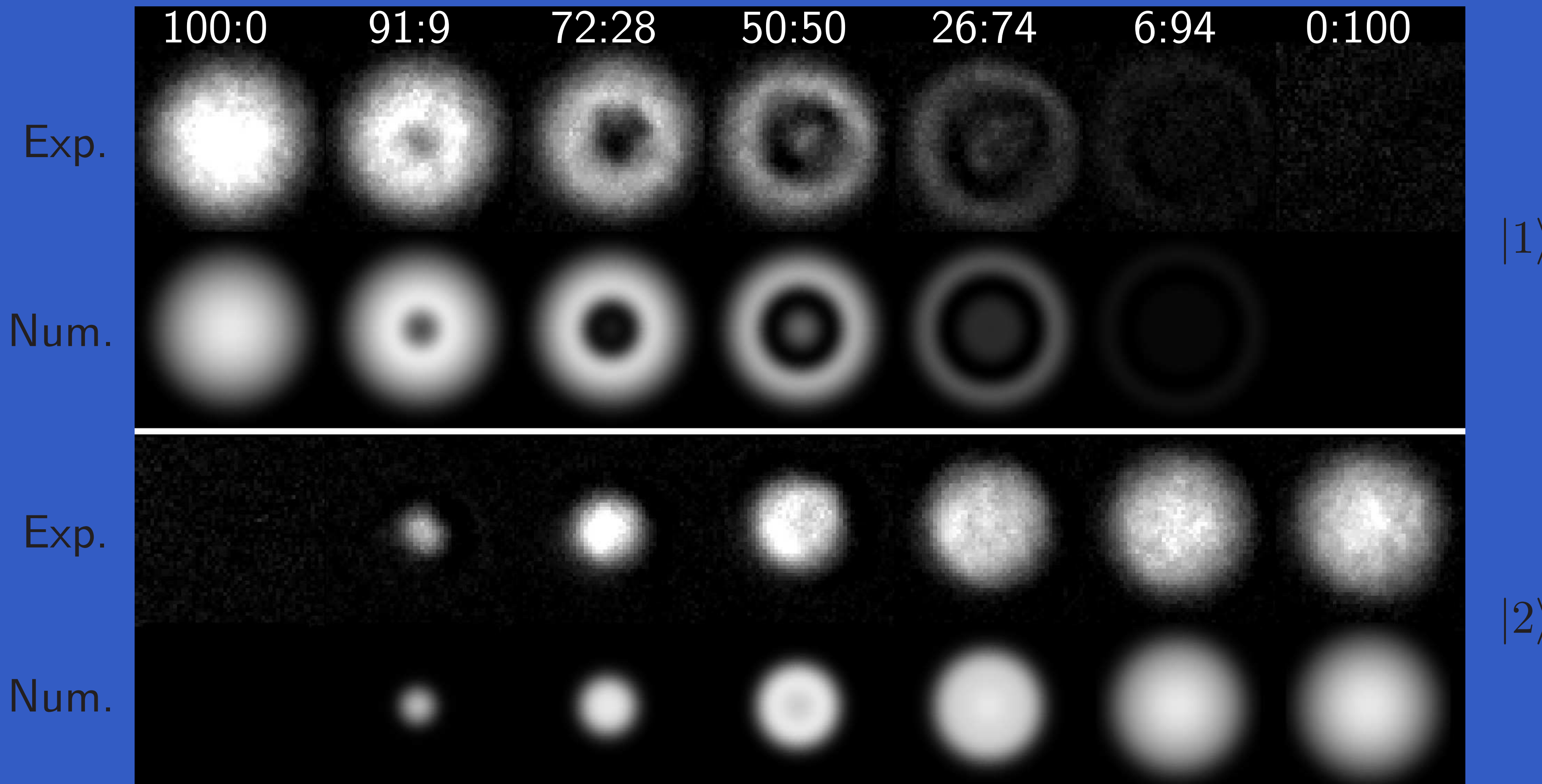
Varying the number of atoms (skip?)

- 50:50 mix for different number of atoms (in thousands).



Varying the atom ratio (skip?)

- Different atoms ratios of $N = 350$K atoms.



Dark-bright oscillations

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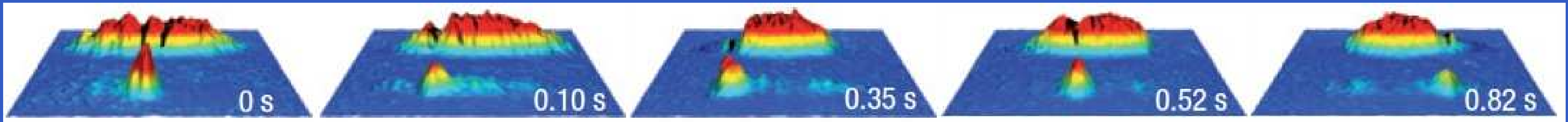
- Dark-bright oscs. [Sengstock's group, Nat. Phys. **4** (2008) 496]:



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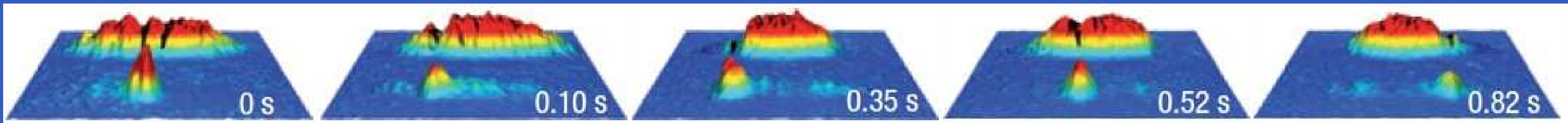


- One-dimensional theory (without a trap):

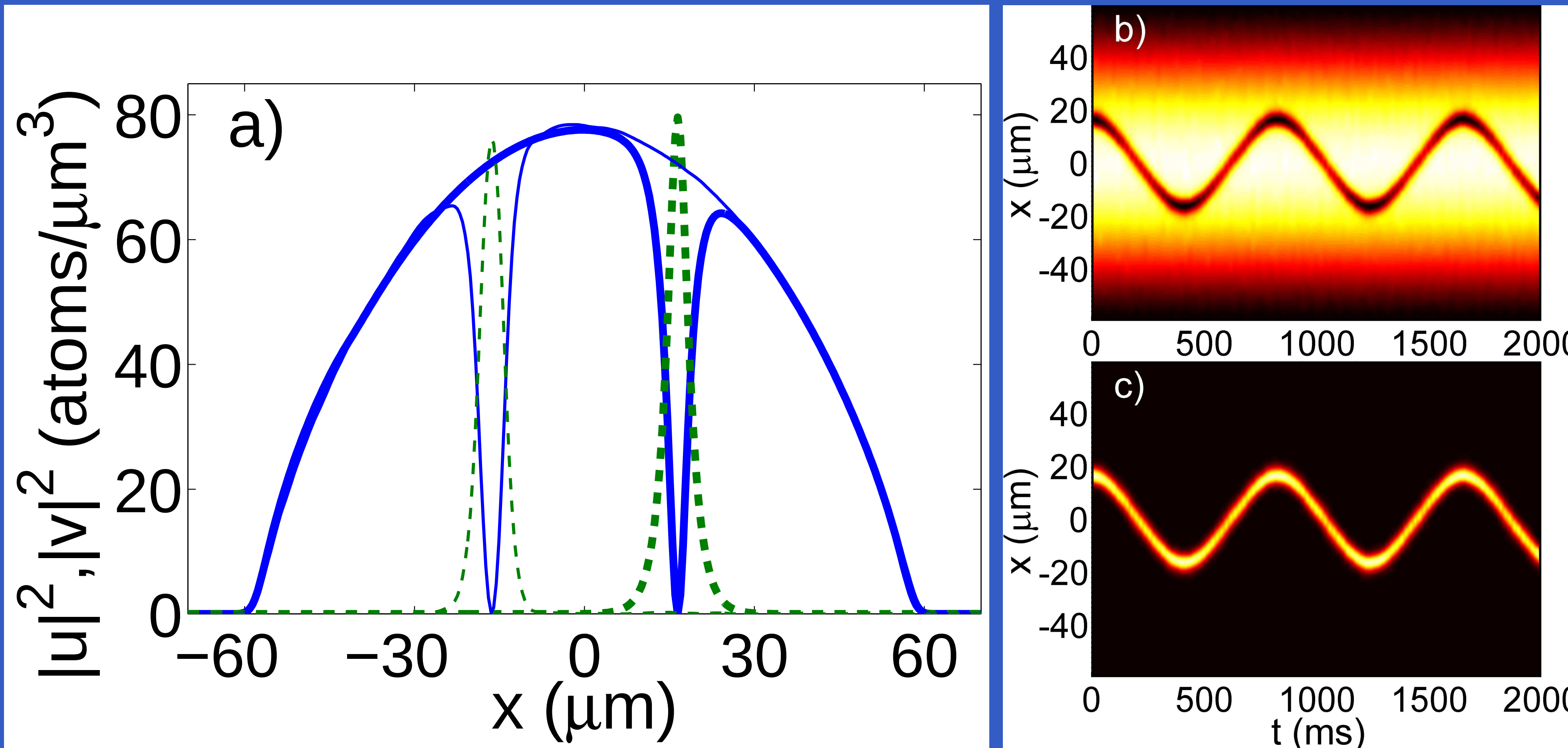
$$\psi_D = \imath\sqrt{\mu}\sin\alpha + \sqrt{\mu}\cos\alpha\tanh(\kappa(x-q(t))),$$

$$\psi_B = \sqrt{\frac{N_B\kappa}{2}}e^{\imath(\phi+\omega_B t + x\kappa\tan\alpha)}\mathrm{sech}(\kappa(x-q(t))).$$

- dark-bright soliton width $\kappa = \sqrt{\mu\cos^2\alpha + (N_B/4)^2} - N_B/4$
- dark-bright soliton position $q(t) = q(0) + t\kappa\tan\alpha$ and phase angle α

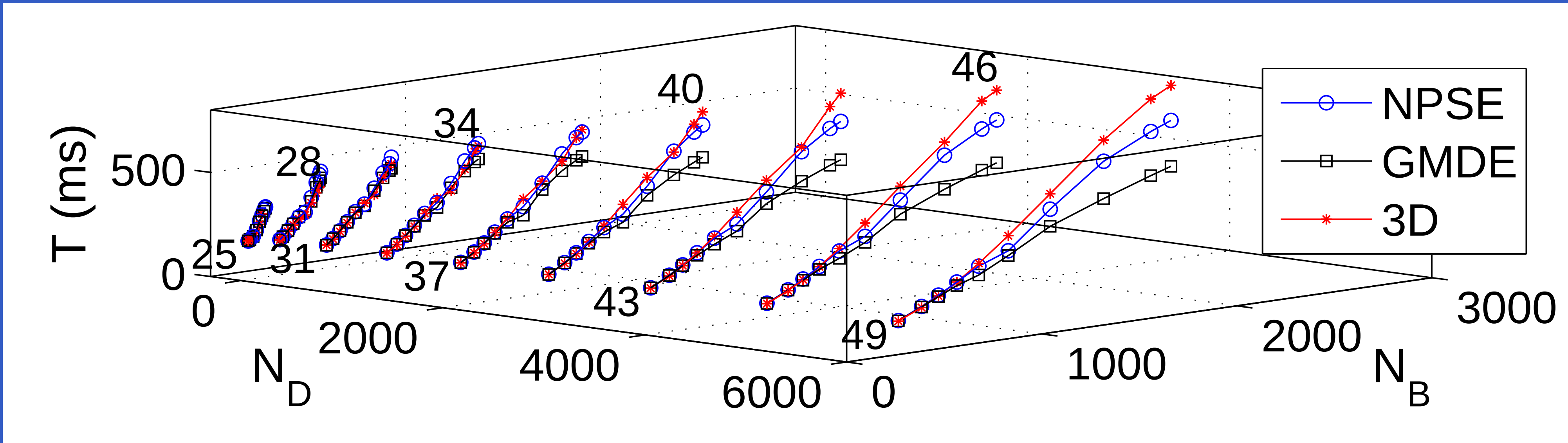
Dark-bright oscillations in 1D

- Dark-bright oscillations in 1D inside a magnetic (parabolic) trap:



Dark-bright oscillations: 1D vs. 3D

- Dark-bright oscillations 1D vs 3D: Oscillation period T (ms)



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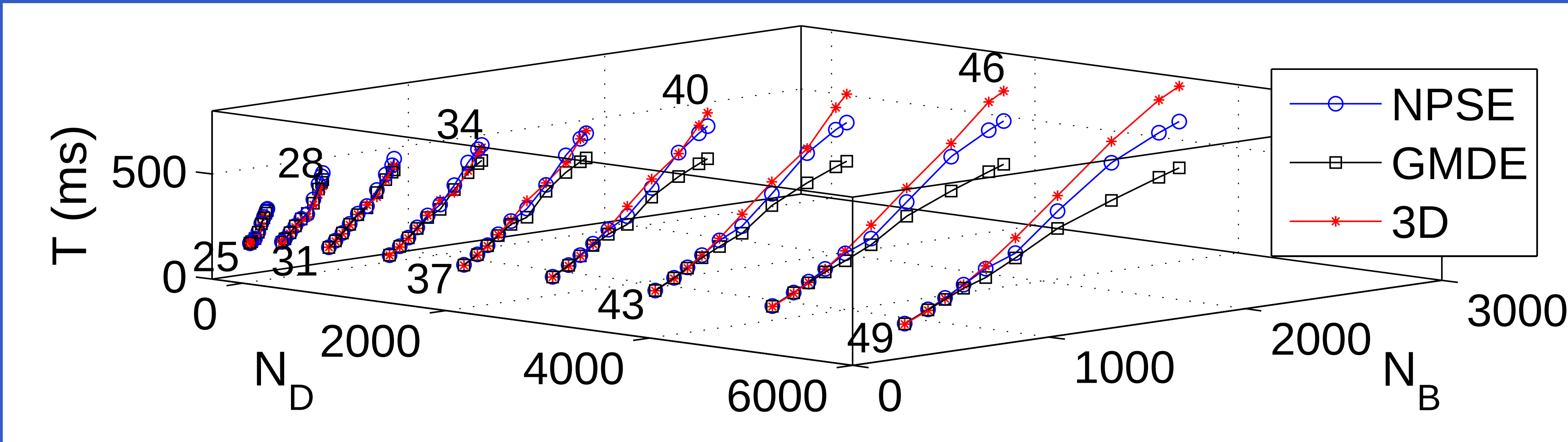
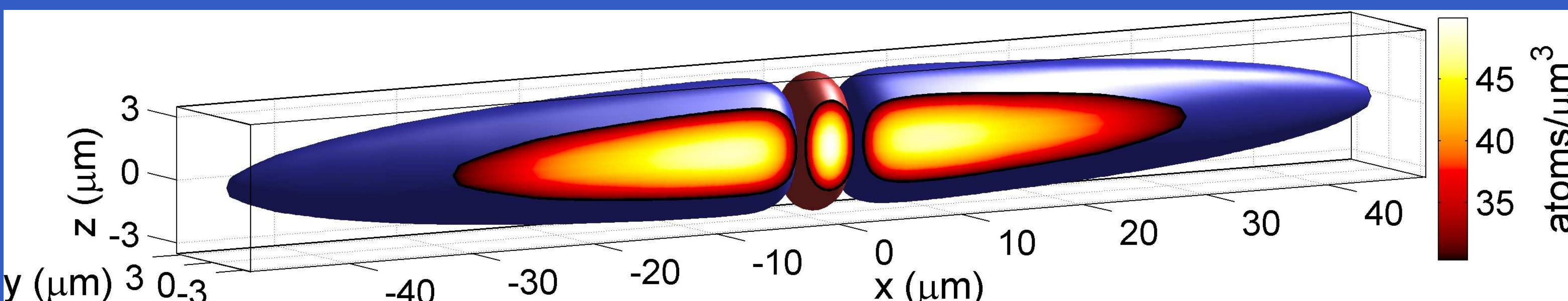
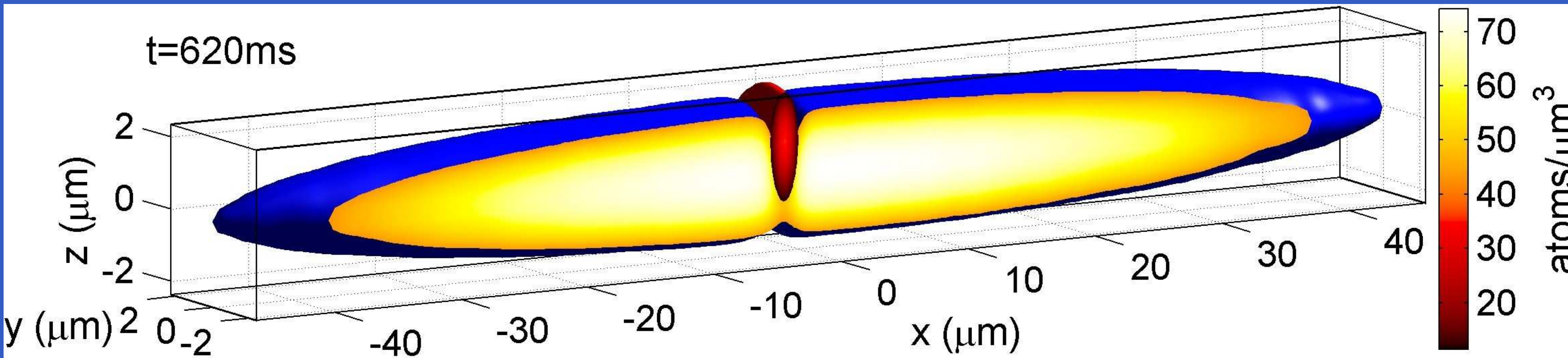


- Largish bright atom number ($N_B \approx N_D/10$) [movie]



Dark-bright oscillations in 3D: numerics

- Smallish bright atom number ($N_B \approx N_D/100$) [movie]



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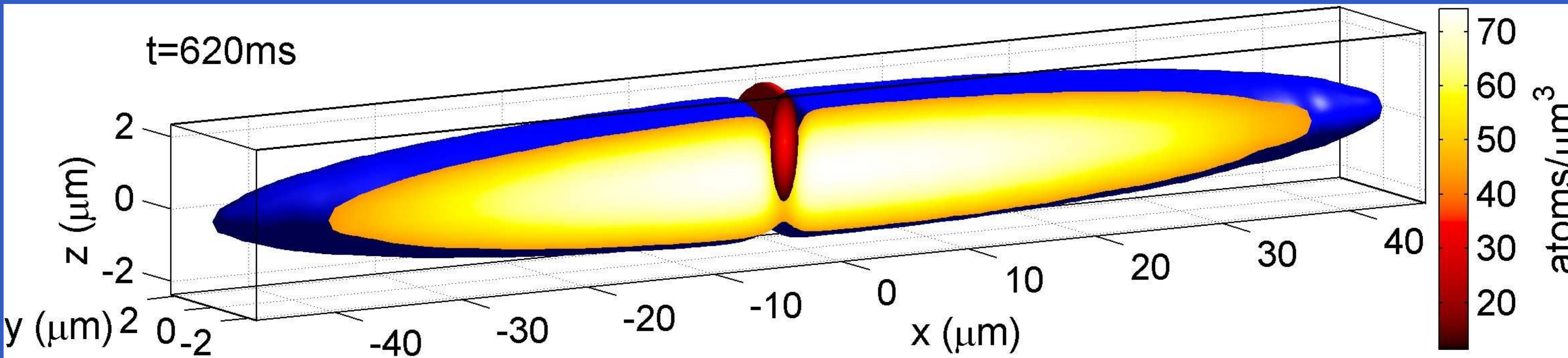
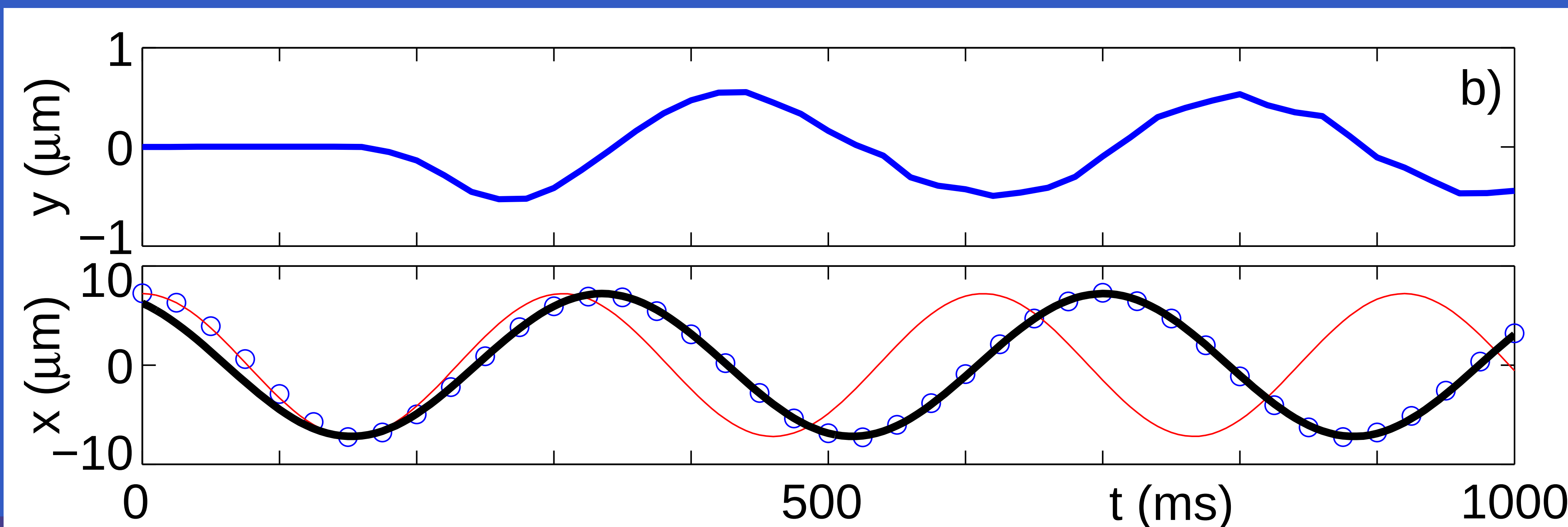


- Center of mass oscillations:



Dark-bright oscillations in 3D: DB-DB interactions

- In DB-DB interactions there are several factors:
  - Dark inside parabolic trap $\Rightarrow$ pull towards center
  - Bright inside parabolic trap $\Rightarrow$ pull towards center
  - Dark-Dark always repulsive
  - Bright-Bright interaction: depends on relative phase:
    - Same phase brights $\Rightarrow$ repulsion
    - Opposite phase brights $\Rightarrow$ attraction

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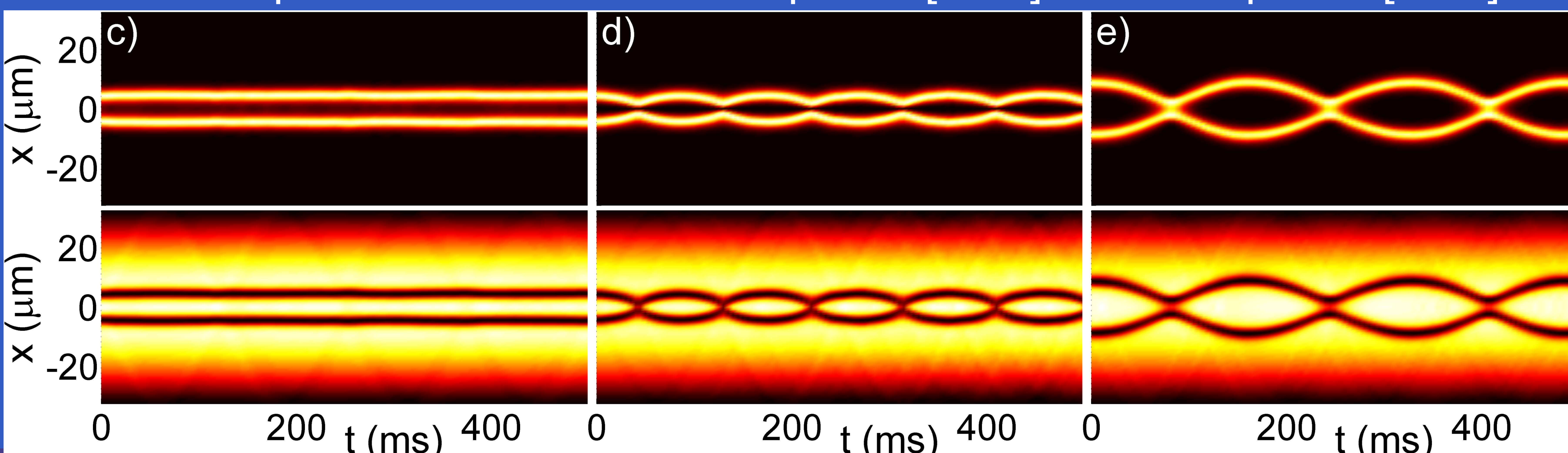
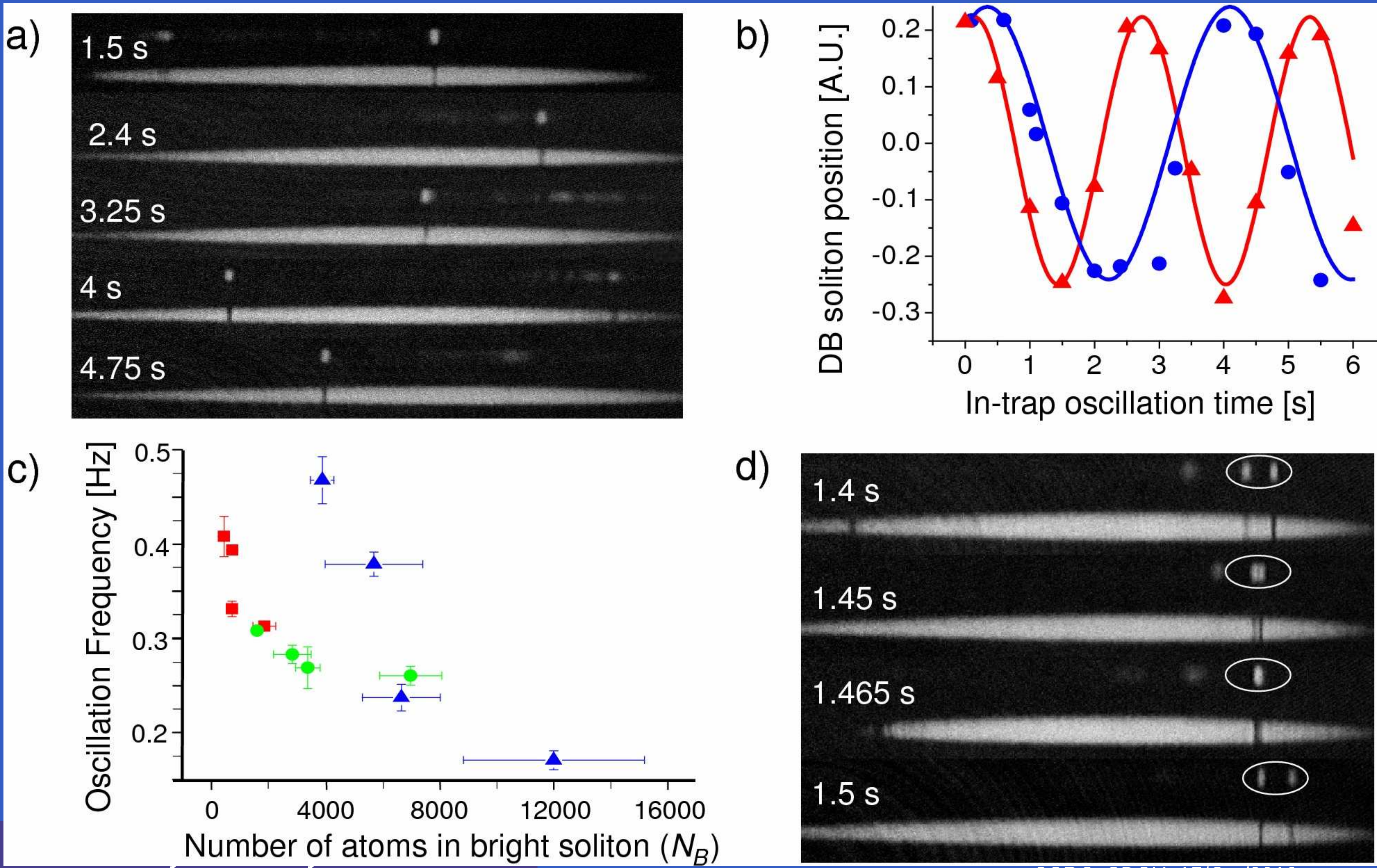
- Different cases for in-phase or out-of-phase brights:

In phase | Out-of-phase [movie] | In-phase [movie]



Dark-bright oscillations: real experiments

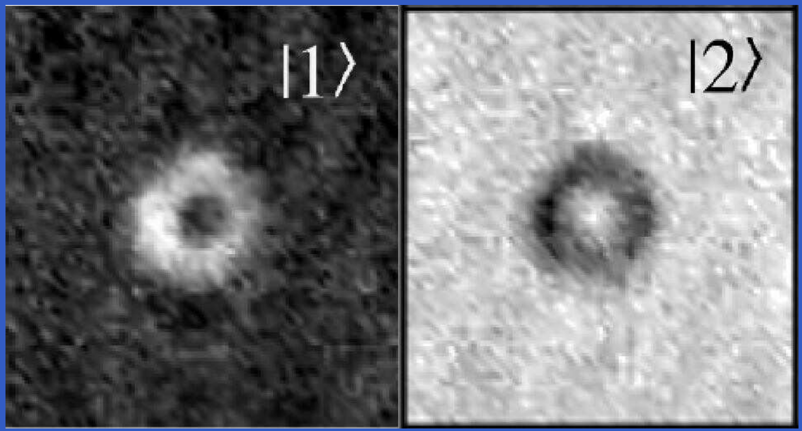
- Peter Engels (Washington State University) experiments:



Vortex-vortex interactions

Repulsive BEC in 2D: Vortex → vortex lattice

- 2 species coupling (1 vortex)
  Matthews, *et al.* PRL **83** (1999) 2498.



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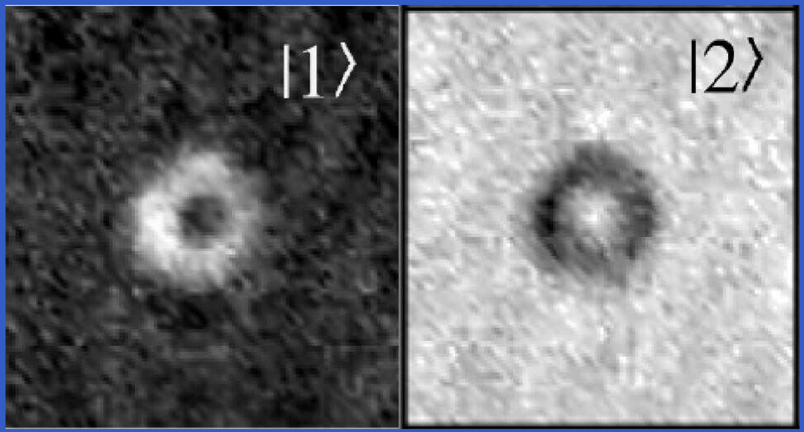
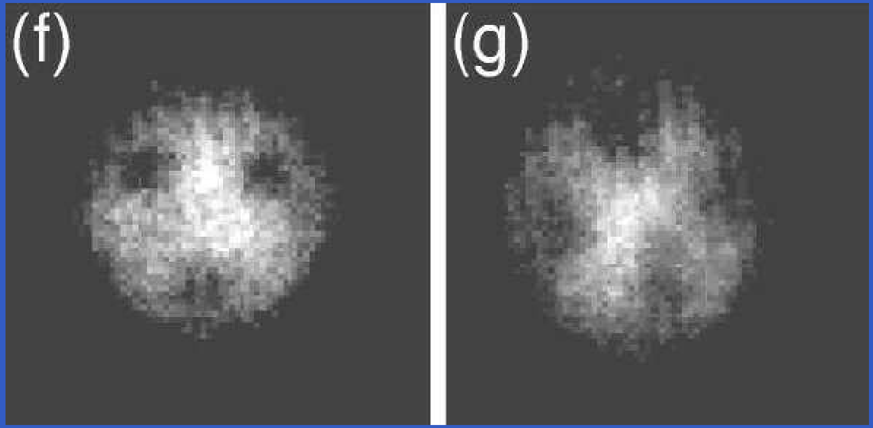


- Stirred with laser (1-4 vortices)
  Madison, *et al.* PRL **84** (2000) 806.



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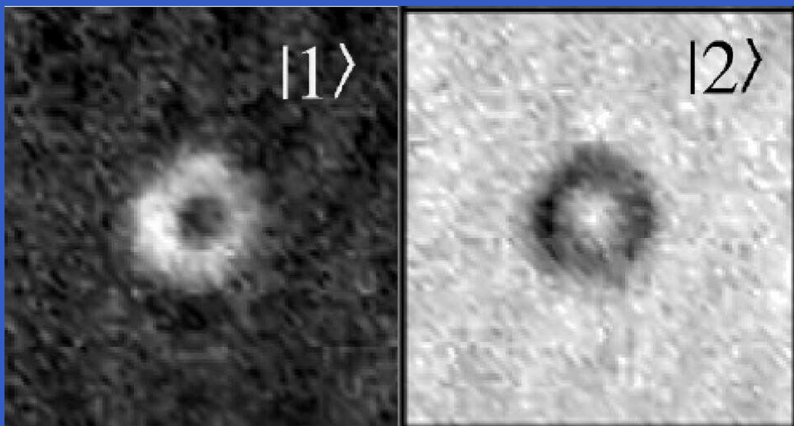
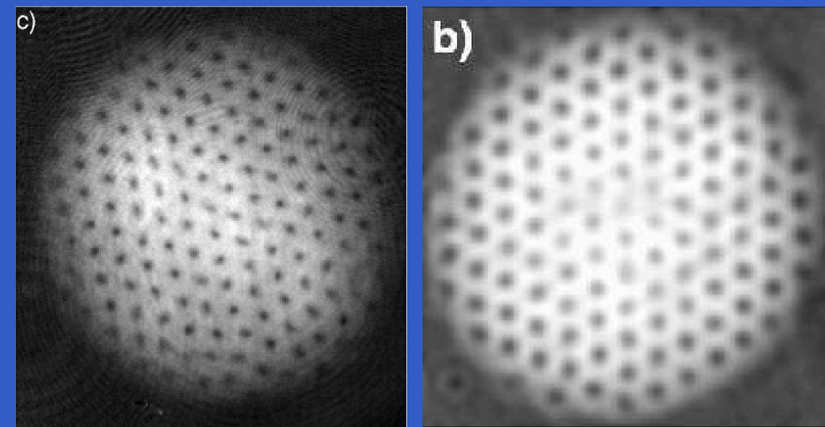


- Stirred (up to 120 vortices)
  Raman, *et al.* PRL **87** (2001) 210402.



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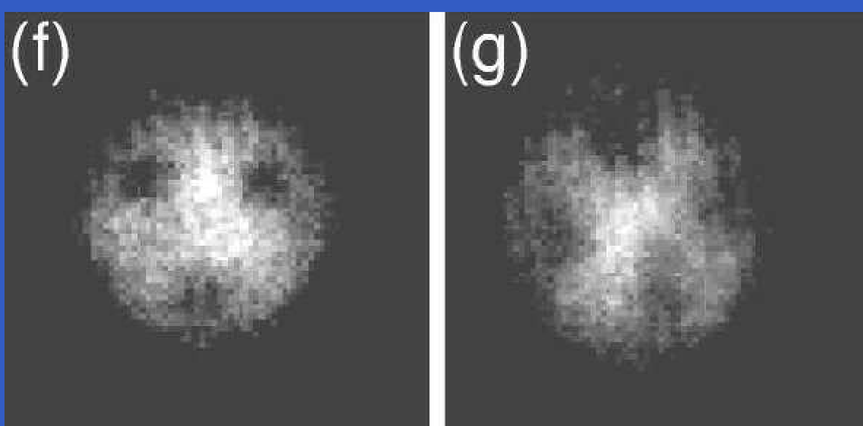

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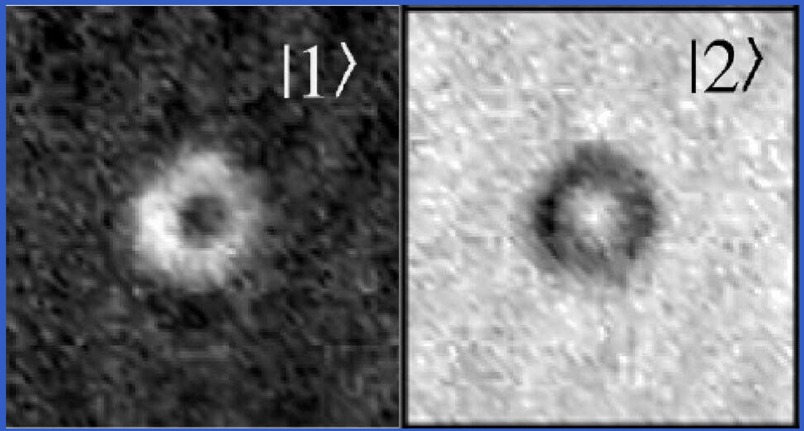

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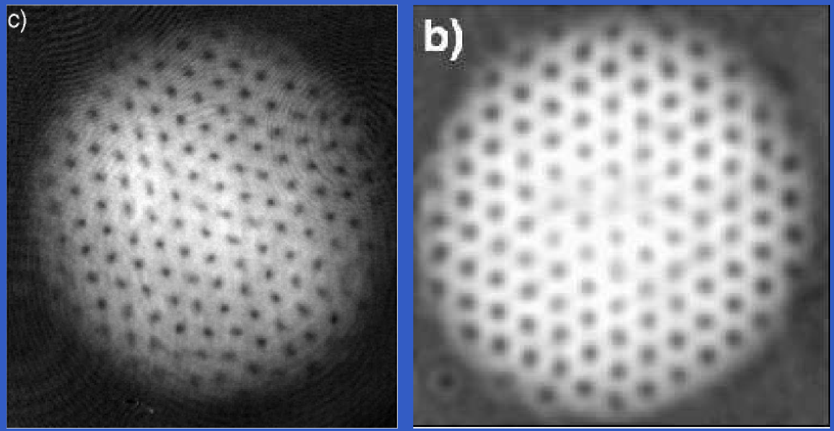

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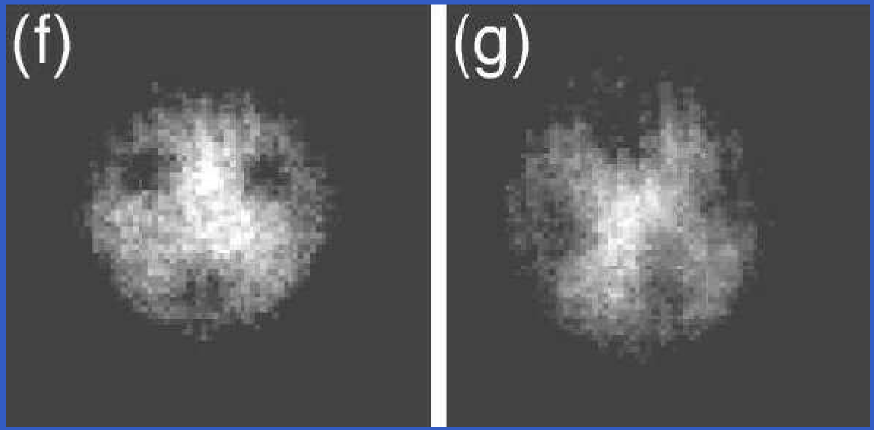
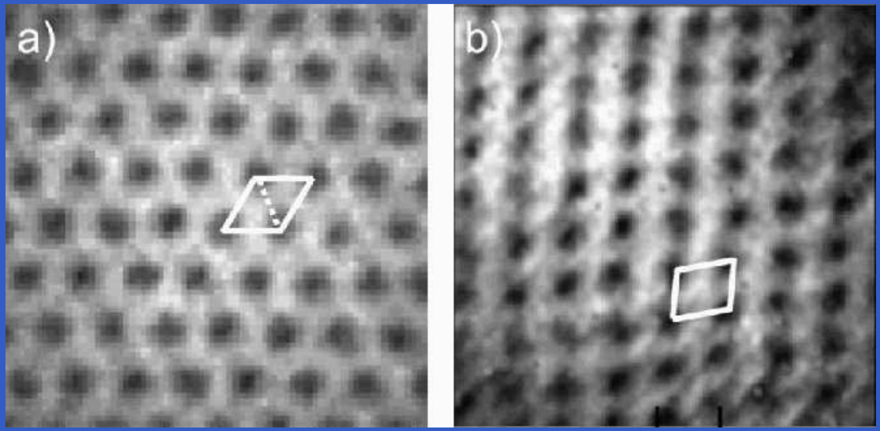
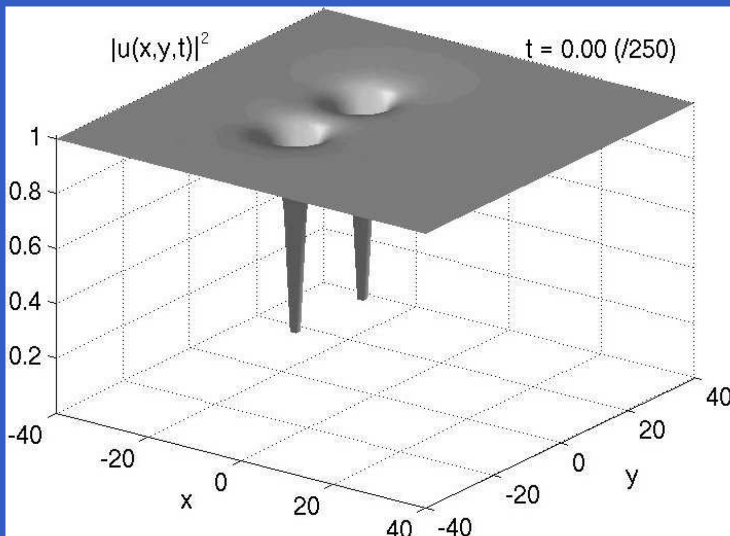


- Compress → deformed VL
  Engels, *et al.* PRL **89** (2002) 100403.



Vortex interactions: pair dynamics (GPE)

- Opposite charge



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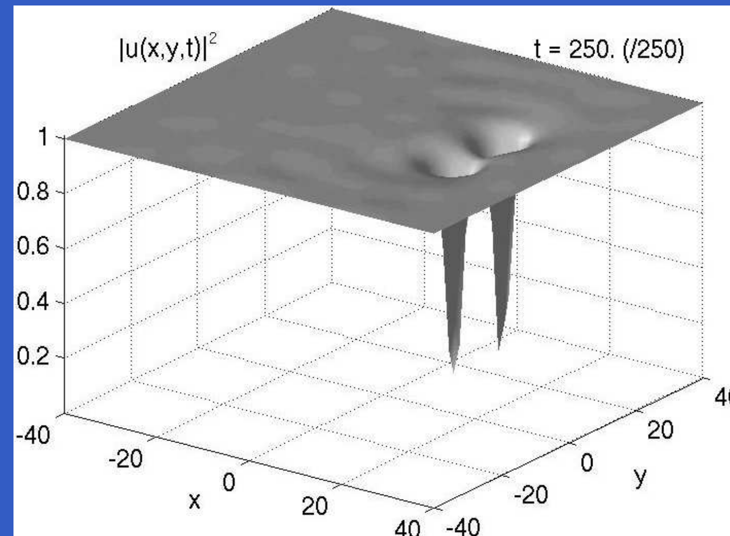

[movie]

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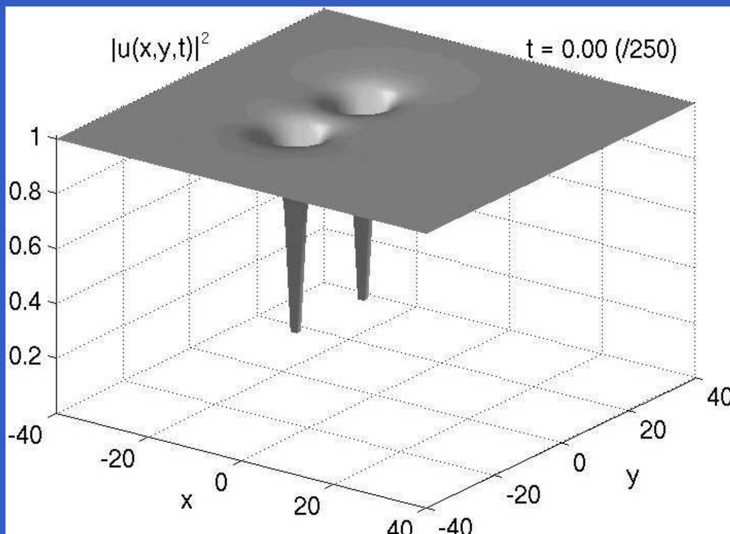

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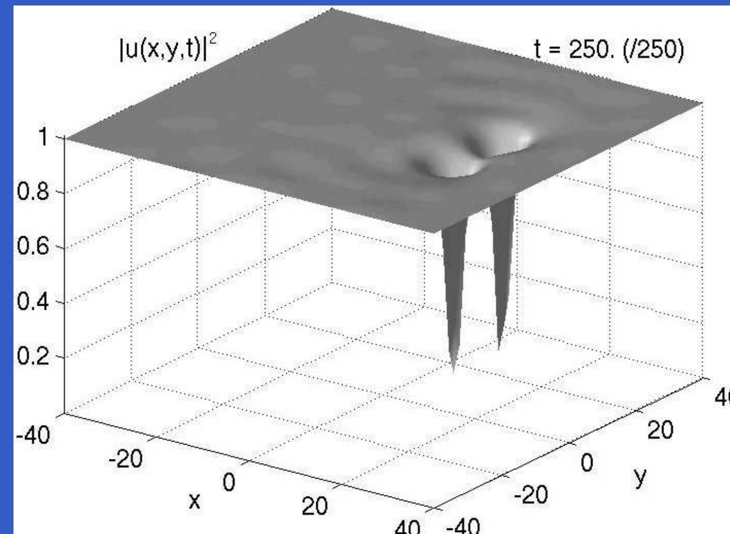
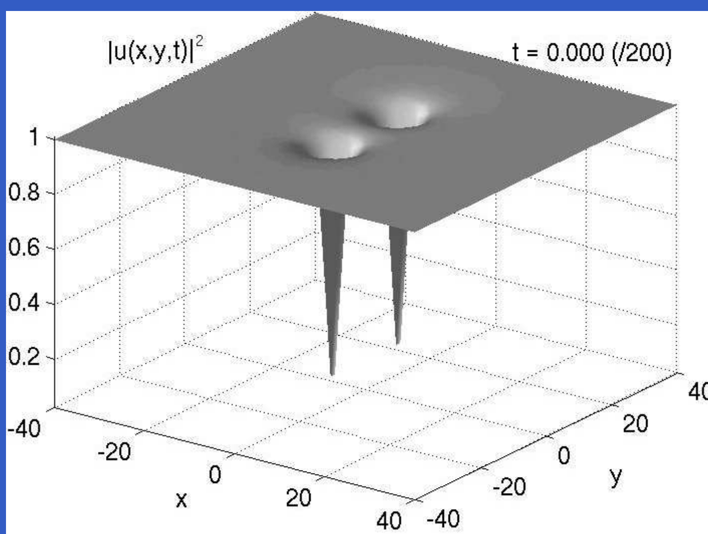


[movie]

- Same charge



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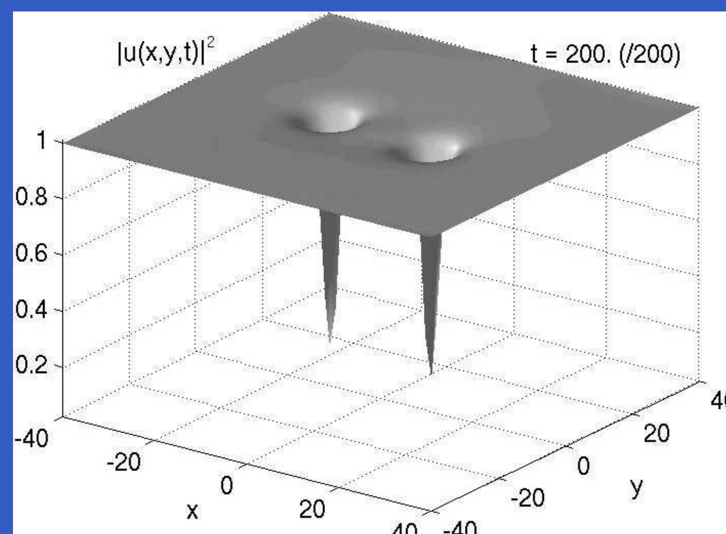
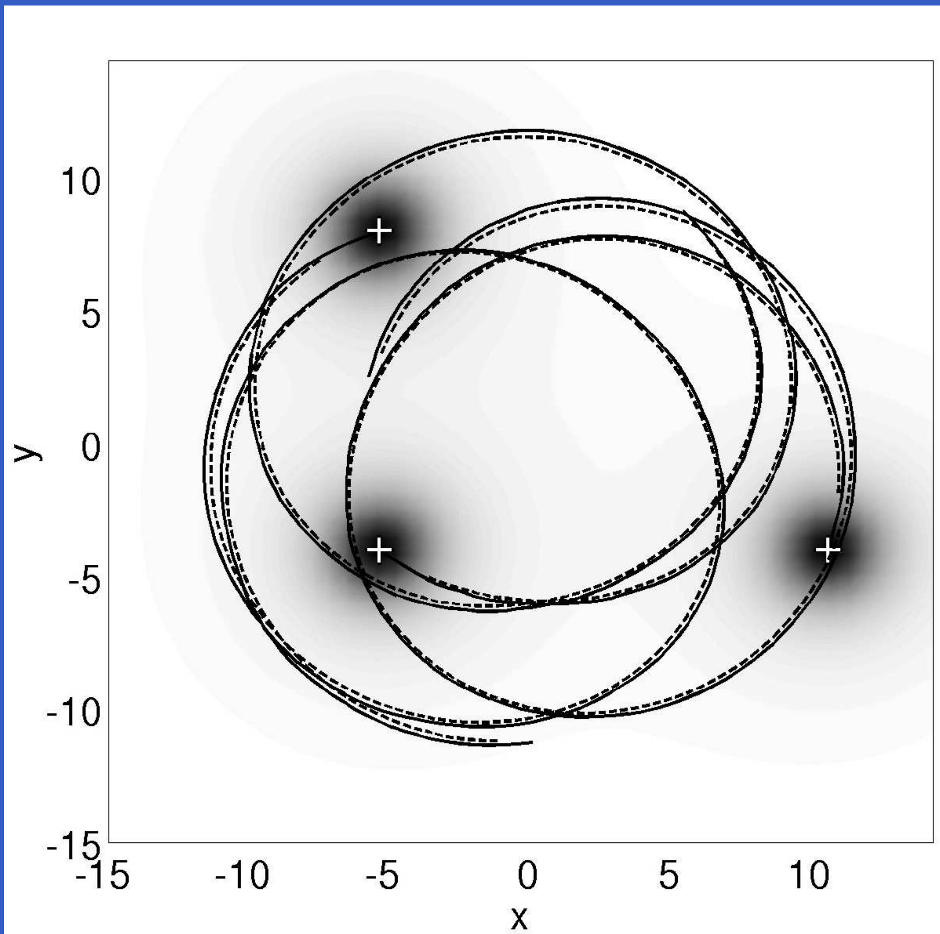
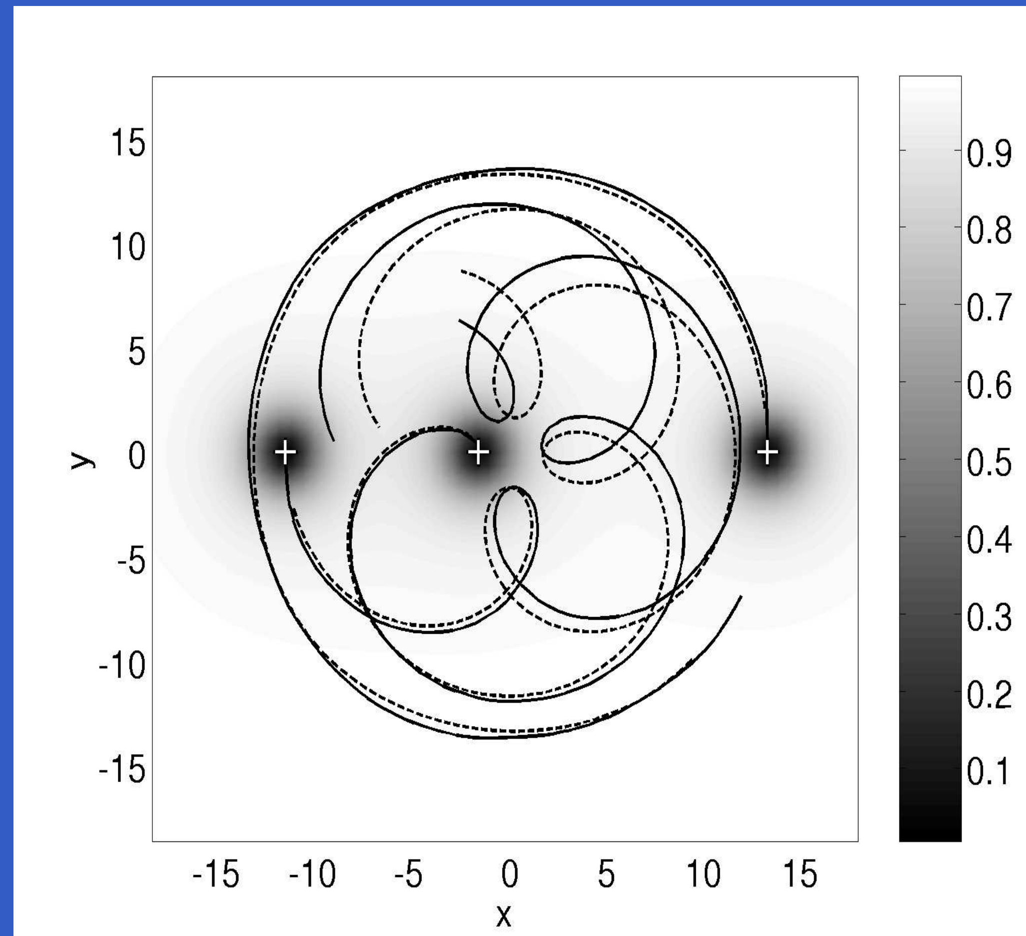


[movie]

Vortex trajectories: PDE vs ODE





[movie] [movie]

Molecular Dynamics (a la Parrinello-Rahman)

- Study an infinite system of vortices (neglect boundary effects)

$\Rightarrow$ Molecular Dynamics simulation

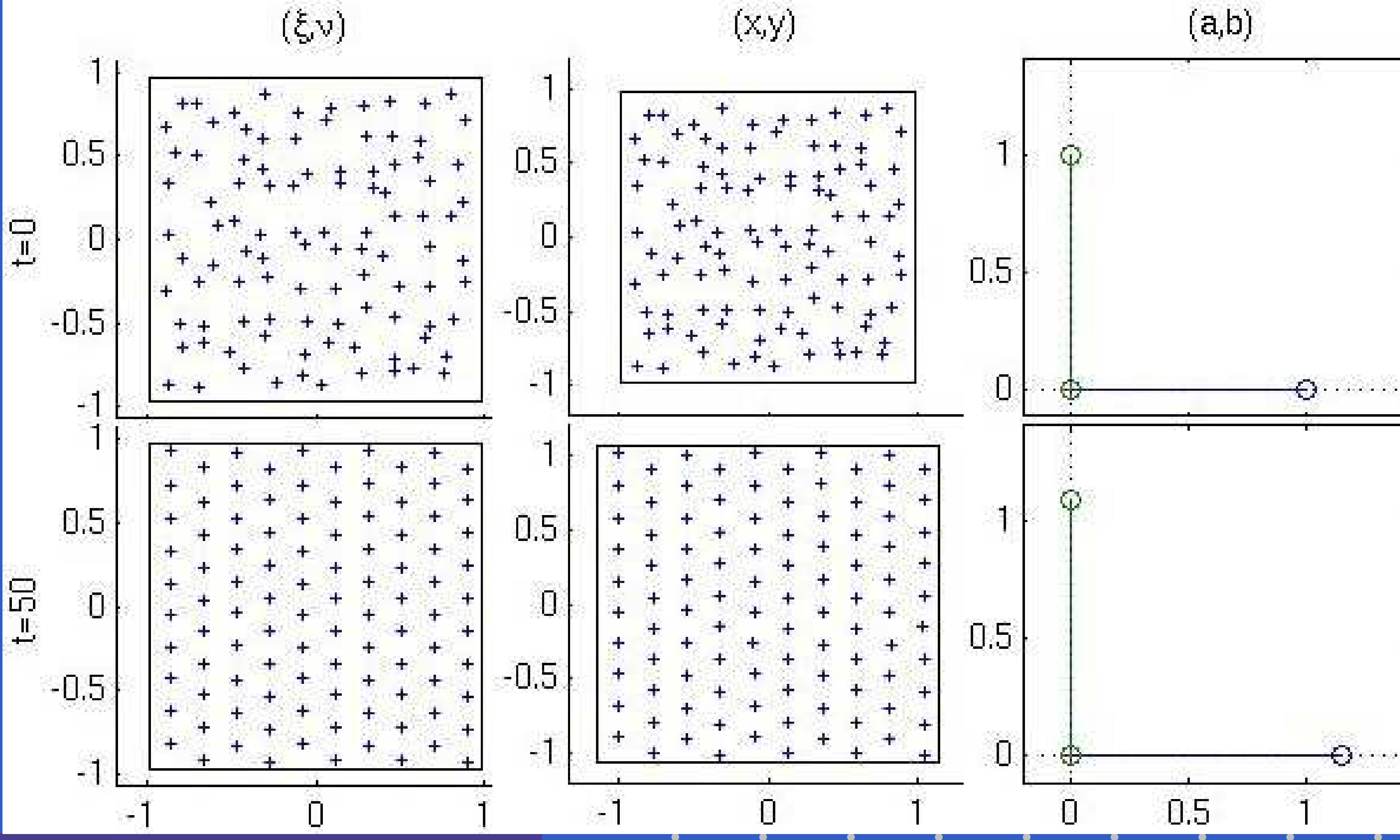
- Take the pairwise potentials $V(r_{ij}, \varphi_{ij})$ and couple all possible pairs
- Introduce a deformable coordinate system (the box) $h = (\vec{a}(t), \vec{b}(t))$
- Introduce inertia of the box (mass=W)
- couple box to particles
- Lagrangian in box coordinates $s_i^\dagger = (\xi_i, \nu_i)$ [i.e. $(x_i, y_i)^\dagger = h\, s_i$]:

$$L = \frac{1}{2}\sum_i m_i\, \dot{s}_i^\dagger\, G\, \dot{s}_i - \sum_i \sum_{j>i} V(r_{ij}, \varphi_{ij}) + \frac{W}{2}\left(\dot{a}_x^2 + \dot{a}_y^2 + \dot{b}_x^2 + \dot{b}_y^2\right) - P_{ext}\, A.$$

- where: $G = h^\dagger\, h$ | m_i mass | A area (volume) box.
- and first term is particle's kinetic energy in box coordinates.

MD: crystalline structures

- Disordered initial condition: [movie]



Vortex-vortex iterations across components

- Could we do the previous reduction in binary BECs?
- We would like to obtain reduced ODEs that capture the dynamics of vortex-vortex interactions across components:

$$
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i\hbar\frac{\partial\psi_1}{\partial t} &= \left(-\frac{\hbar^2}{2m}\nabla^2 + V_1 + a_{11}\left|\psi_1\right|^2\right)\psi_1 + \left(a_{12}\left|\psi_2\right|^2 + \kappa\psi_2\right)\psi_1, \\
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- Work in progress: Eunsil Baik + Rafael Navarro + Prof. Bromley & Co.

END... Thanks!

NLDS: Nonlinear Dynamical Systems @ SDSU

http://nlds.sdsu.edu/ [Graduate Programs]

MS in Appl. Mathematics with concentration in Dynamical Systems.

- Fall Year 1:
  - MATH-537 : Advanced Ordinary Differential Equations
  - MATH-538 : Dynamical Systems & Chaos I
  - MATH-636 : Mathematical Modeling
- Spring Year 1:
  - MATH-531 : Advanced Partial Differential Equations
  - MATH-639 : Nonlinear Waves
  - MATH-638 : Dynamical Systems & Chaos II
- Fall Year 2:
  - MATH-635 : Pattern Formation
  - MATH-693A : Advanced Numerical Analysis
  - MATH-797 : Research
- Spring Year 2:
  - MATH-799A : Thesis – Project