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Mimetic Divergence, Gradient, Curl and Boundary Operators over Non-uniform, Two Dimensional Meshes

David Batista

Advisors: **Dr. Jose Castillo** **Gustaaf Jacobs**

Agenda

- The general idea
- Discretization of the domain – space discretization
- Discrete mimetic operators over non-uniform meshes
- Numerical implementation
  - Robin boundary conditions for general curvilinear meshes
  - Automatic feature adaptation
  - 4^{th} order solution. Why curvilinear meshes
  - Full tensor problem
- Conclusions
- Thank you

General idea

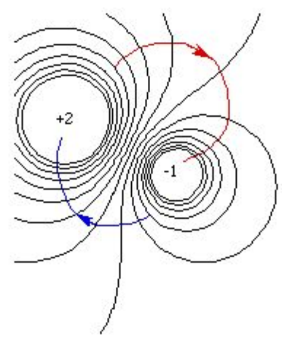
Partial Differential Equations (PDE's)

Poisson Eqn. $div\left(grad\ g\right) = \rho$

Heat Eqn. $\frac{\partial T}{\partial t} - \alpha\ div\left(grad\ T\right) = 0$

Wave Eqn. $\frac{\partial^2 f}{\partial t^2} - div\left(grad\ f\right) = 0$

Maxwell Eqn. $\frac{\partial \vec{B}}{\partial t} + curl\left(\vec{E}\right) = 0$



Physical Phenomena

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General idea

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Methods for solving PDE's

Conservative
Versatile (different types of domains)

Easy to implement
Easy to understand

Finite Difference — **MIMETIC OPERATORS** — Finite Element / Finite Volume

May not be conservative

Not easy implementation

Methods for solving PDE's

MIMETIC OPERATORS

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Versatile (different types of domains)

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May no

entation

Numerical method for solving PDE's.
Easy to implement.
Conservative.

Space discretization

Finite Difference

Finite Element
Finite Volume

1D	uniform	Non-uniform	structured	unstructured
2D	uniform	Non-uniform	structured	unstructured
3D	uniform	Non-uniform	structured	unstructured

Spatial dimension

Meshes

1D	uniform	Non-uniform	structured	unstructured
2D	uniform	Non-uniform	structured	unstructured
3D	uniform	Non-uniform	structured	unstructured

MIMETIC
OPERATORS

WORK IN PROGRESS

Methods for solving PDE's

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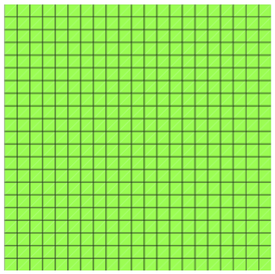
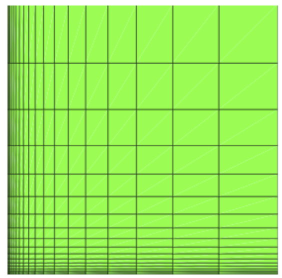
Easy to implement.
Conservative.
Work in progress.

Space discretization

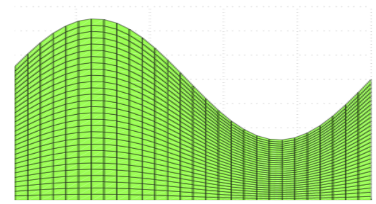
Where to solve PDE's



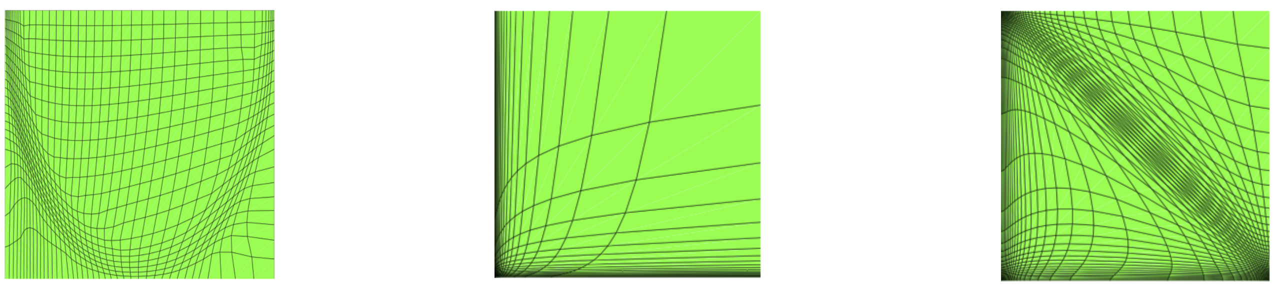
Cartesian grid



Non-uniform tensor product mesh



Curvilinear Boundaries



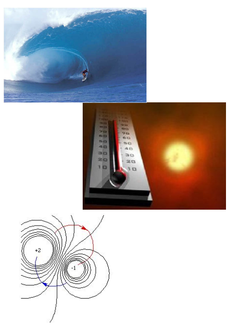
General curvilinear mesh

Discrete Operators

Spatial dimension	Meshes
Continuum	**Discrete**

Discrete Operators

Spatial dimension	Meshes
Continuum	**Discrete**



$$div\,(grad\; g) = \rho$$

$$\frac{\partial^2 f}{\partial t^2} - div\,(grad\; f) = 0$$

$$\frac{\partial T}{\partial t} - \alpha\, div\,(grad\; T) = 0$$

$$\frac{\partial \vec{B}}{\partial t} + curl\,(\vec{E}) = 0$$

Continuum	Discrete
div	D
grad	G
curl	C

Mimetic Operators

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High order, discrete differential operators
Divergence, Gradient, and Curl.

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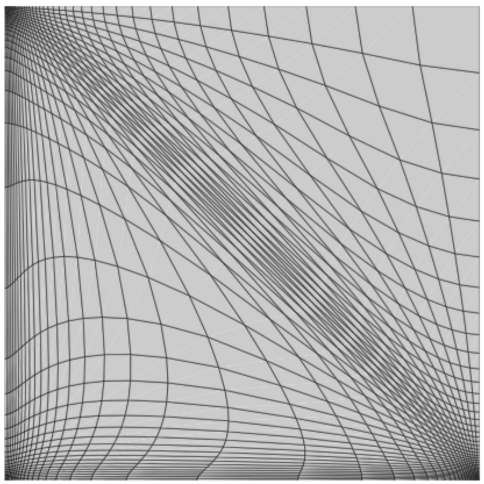
Discrete Operators

D G C ← Discrete Analogs of continuous differential operators



Local transformations of cells
1D mimetic operators
Stokes' theorem
Flux across the boundary of the cells
Linear algebra
Theorem of work
Freestream preservation

$$\langle \mathbf{D}\, v,\ f \rangle_Q + \langle v,\ \mathbf{G}\, f \rangle_P = \langle \mathbf{B}\, v,\ f \rangle_I$$ ← Discrete Conservation Law

Discrete Operators

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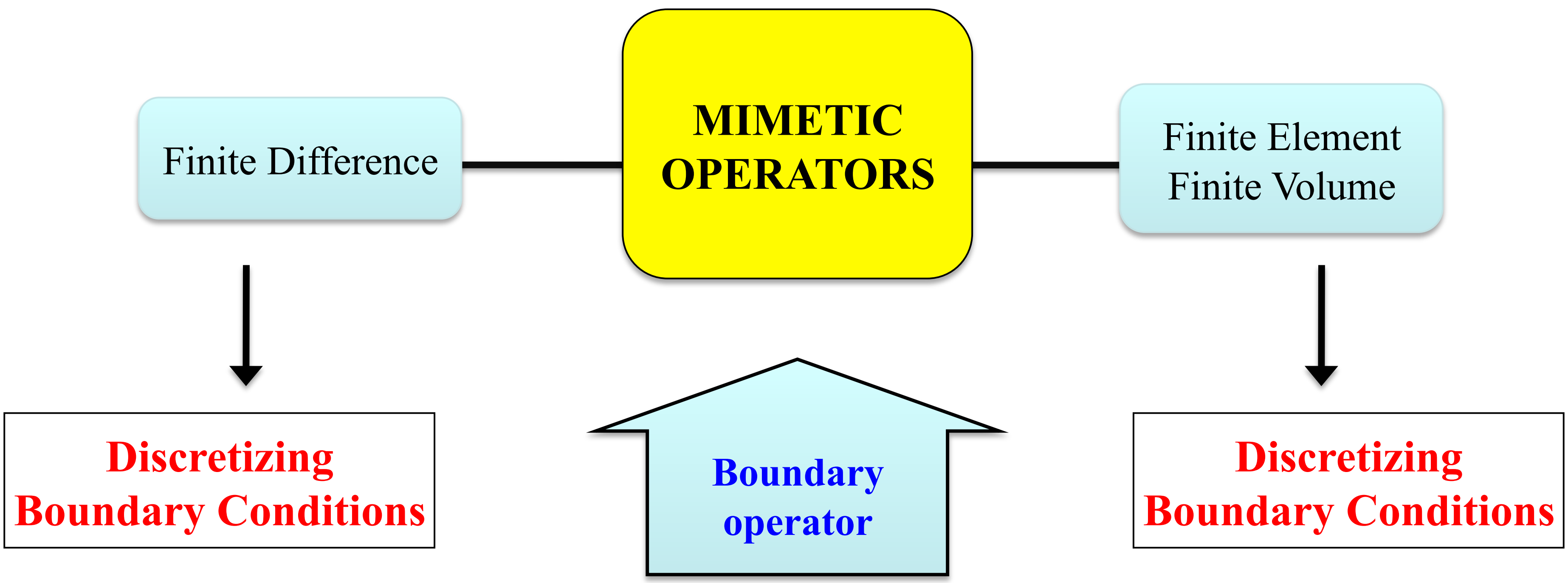
PDE

$$\frac{\partial^2 f}{\partial t^2}(\vec{x},t) - div\left(grad\ f(\vec{x},t)\right) = 0 \quad .... \quad \vec{x} \in \Omega$$

Boundary conditions

$$\alpha f(\vec{x},t) + \beta_1 \frac{\partial f}{\partial n}(\vec{x},t) = g_1(\vec{x},t) \quad .... \quad \vec{x} \in \partial\Omega$$



$$\langle \mathbf{D}\, v,\ f \rangle_Q + \langle v,\ \mathbf{G}\, f \rangle_P = \langle \mathbf{B}\, v,\ f \rangle_I$$

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High order, discrete differential operators
Divergence, Gradient, and Curl.
Unique Boundary operator.

Numerical Implementation -- 1

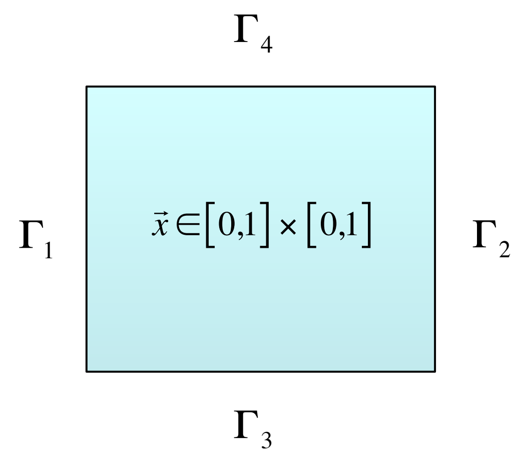
1 – Robin Boundary Conditions and General Curvilinear Mesh

Numerical Implementation -- 1

$$-div\left(grad\left(f\left(\vec{x}\right)\right)\right) = F\left(\vec{x}\right) \longrightarrow \text{PDE}$$

Robin B.C.

$$\begin{cases} \alpha f + \beta_1 \dfrac{\partial f}{\partial n} = g_1(\vec{x}) & \text{............... at } \Gamma_1 \\ \alpha f + \beta_2 \dfrac{\partial f}{\partial n} = g_2(\vec{x}) & \text{............. at } \Gamma_2 \\ \alpha f + \beta_1 \dfrac{\partial f}{\partial n} = g_1(\vec{x}) & \text{............... at } \Gamma_3 \\ \alpha f + \beta_2 \dfrac{\partial f}{\partial n} = g_2(\vec{x}) & \text{............. at } \Gamma_4 \end{cases}$$



$$F\left(\vec{x}\right) = F\left(x,y\right) = \frac{2\times 10^6}{\arctan(100)}\left[\frac{x}{(1+1\times 10^4 x^2)^2} + \frac{y}{(1+1\times 10^4 y^2)^2}\right]$$

$$\alpha = 1, \quad \beta_1 = -\tfrac{1}{1+10^4}, \quad \beta_2 = 1, \quad C = \tfrac{100}{1+10^4}, \quad g_1\left(\vec{x}\right) = \frac{\arctan\left(100\vec{x}\right)+C}{\arctan\left(100\right)}, \quad g_2\left(\vec{x}\right) = 1 + \frac{\arctan\left(100\vec{x}\right)+C}{\arctan\left(100\right)}$$

Numerical Implementation -- 1

Discretizing the problem

$$-div\Big(grad\big(f(\vec{x})\big)\Big) = F(\vec{x}) \longrightarrow \text{PDE}$$

$$-\mathrm{DG}\,\mathrm{f} = \mathrm{F}$$

$$\text{Robin B.C.} \longleftarrow \alpha f + \beta_i \frac{\partial f}{\partial n} = g_i(\vec{x})$$

$$\mathrm{A_D}\,\mathrm{f} + \mathrm{A_N}\,\mathrm{BG}\,\mathrm{f} = \mathrm{b}$$

$$\left(\mathrm{A_D} + \mathrm{A_N}\,\mathrm{BG} - \mathrm{DG}\right)\mathrm{f} = \mathrm{F} + \mathrm{b}$$

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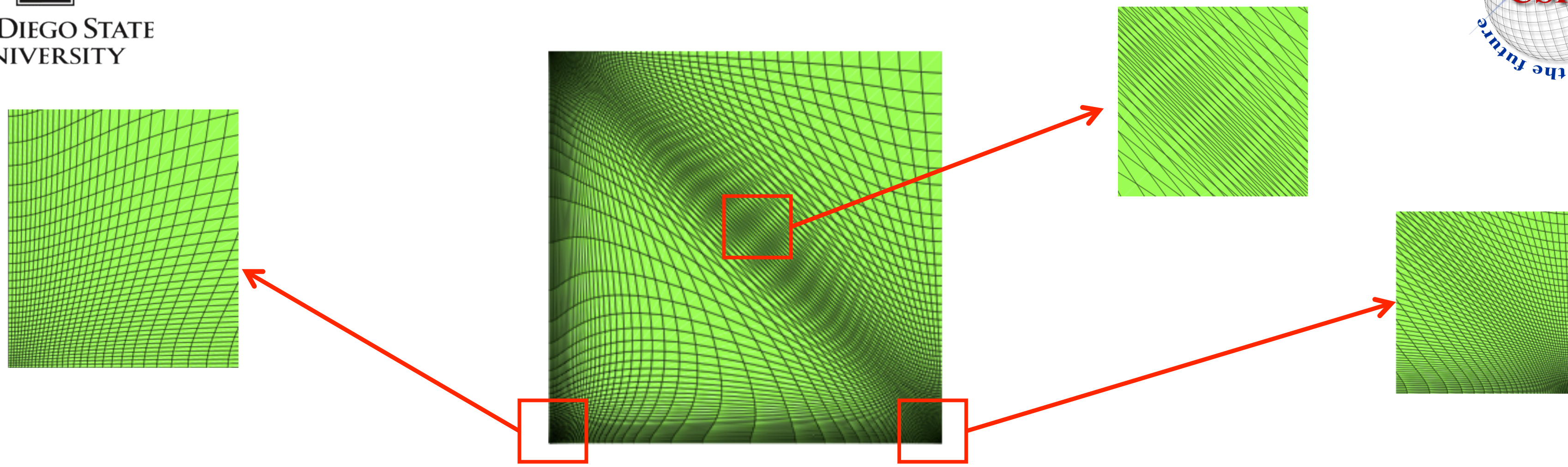
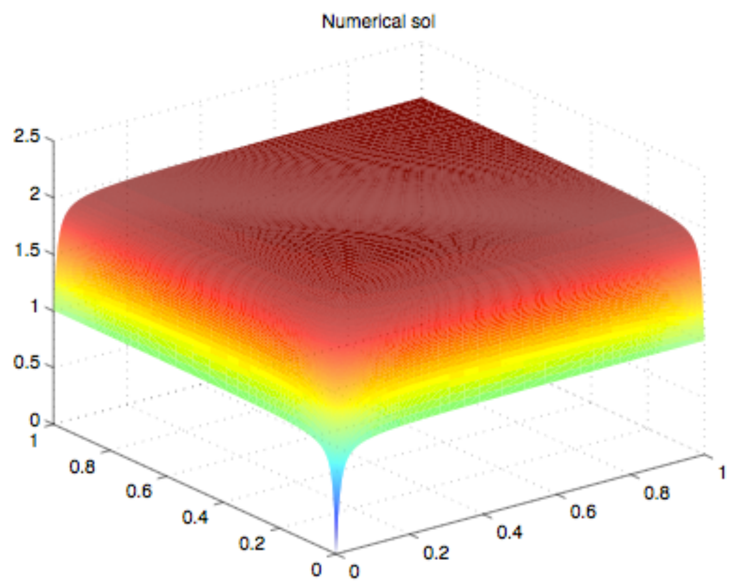
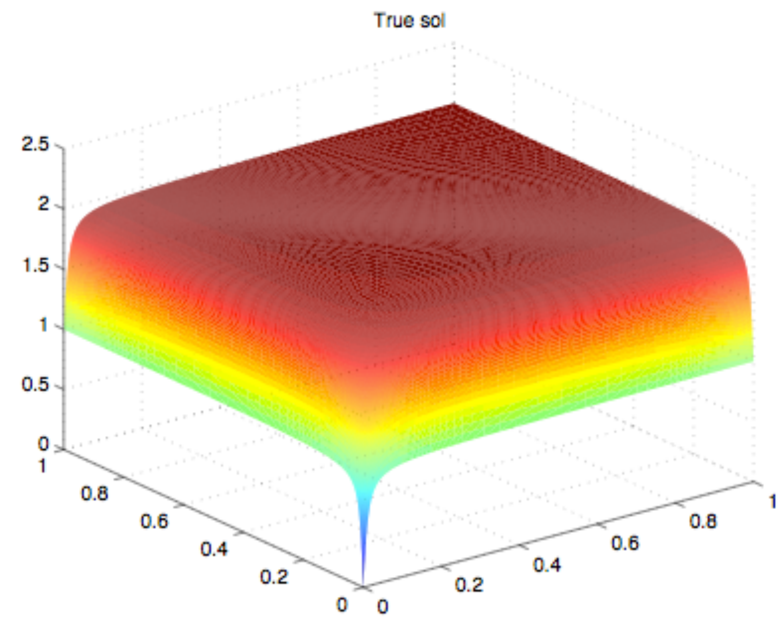
Numerical Implementation -- 1



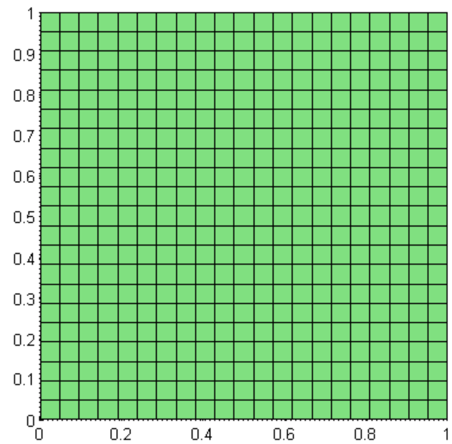


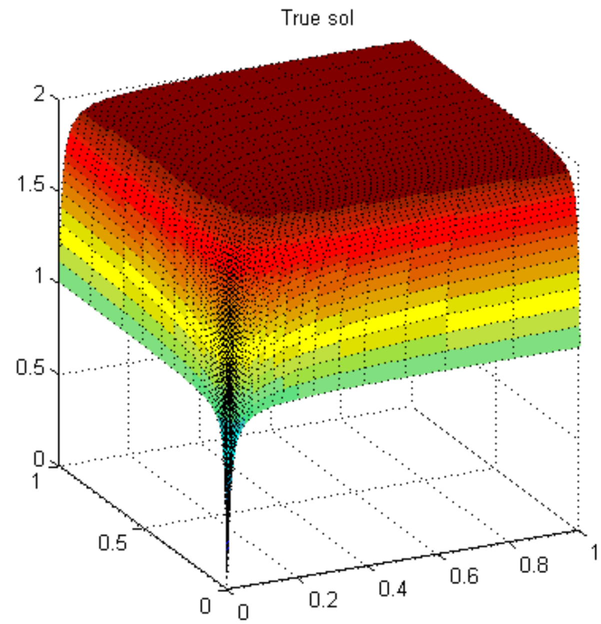


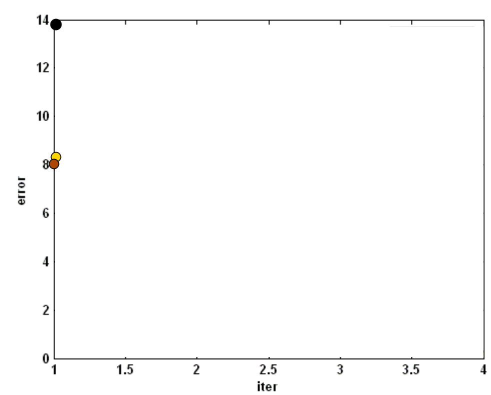
Numerical Implementation -- 2

2 – Automatic Feature Adaptation

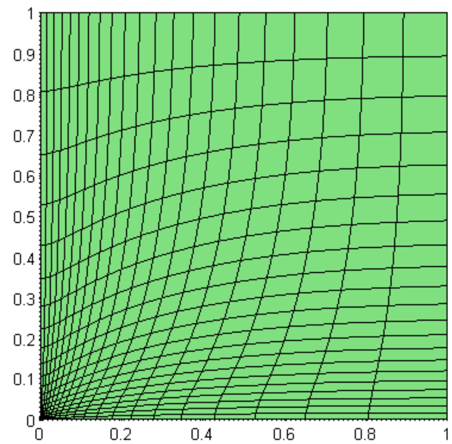
Numerical Implementation -- 2

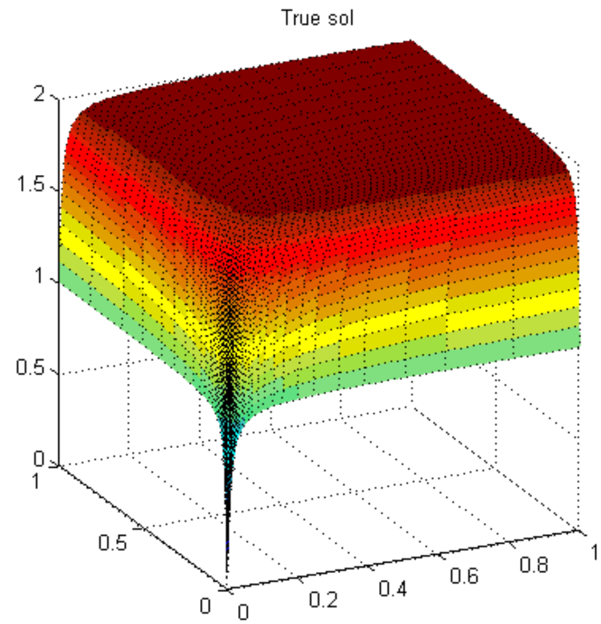


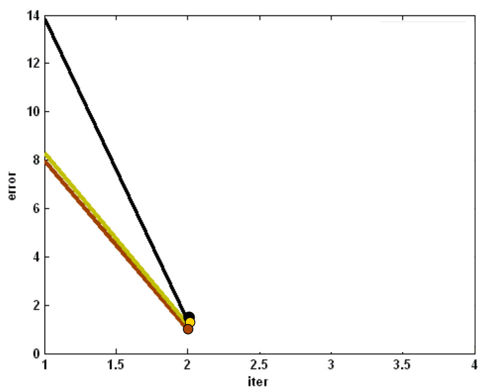




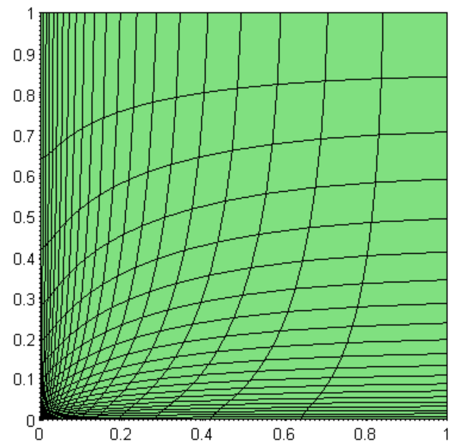
Numerical Implementation -- 2

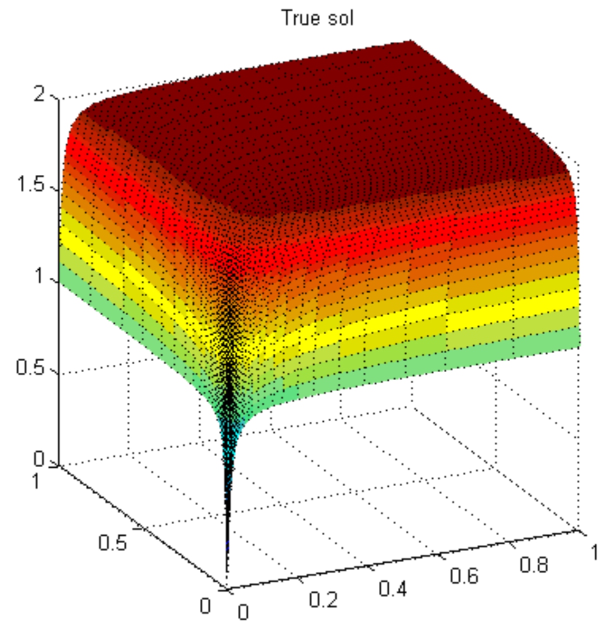


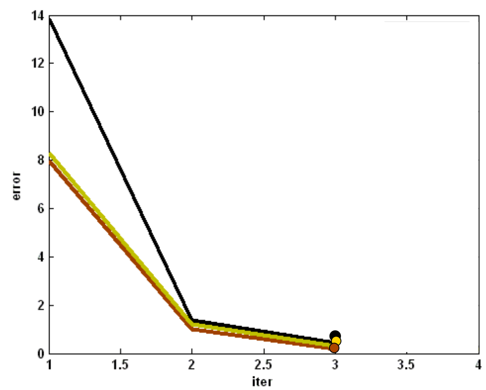




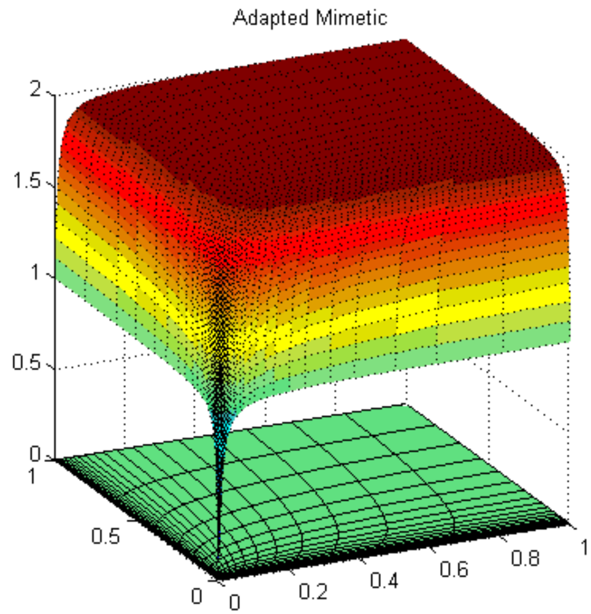
Numerical Implementation -- 2

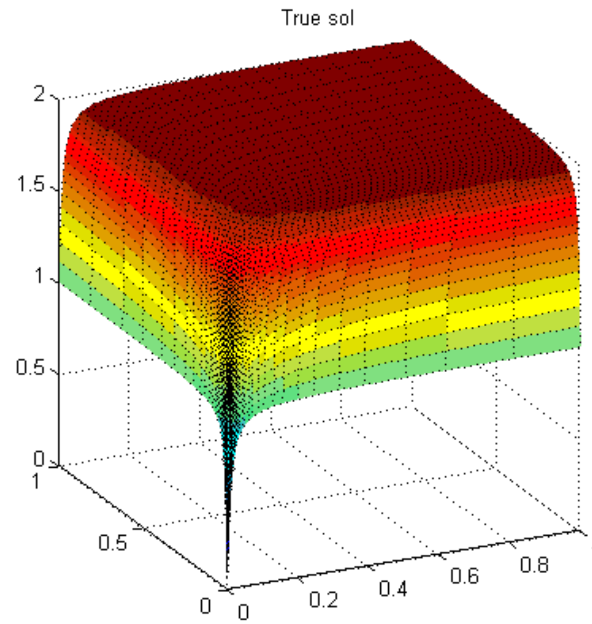


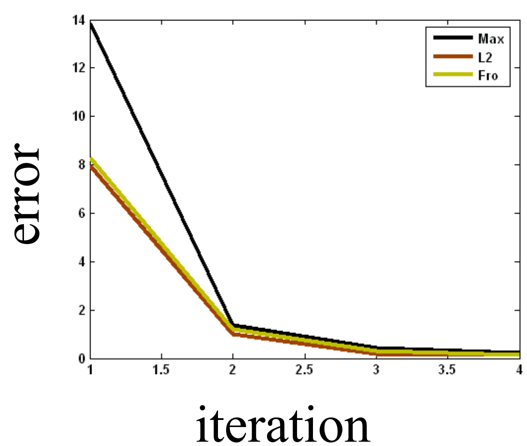




Numerical Implementation -- 2







Mesh size function

$$\frac{1}{\left\| \mathrm{G\,f} \right\| + a} + b$$

Numerical Implementation -- 3

3 – 4th order approximation.
Why curvilinear meshes.

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Numerical Implementation -- 3

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$$-div\left(grad\left(f\left(x\right)\right)\right) = F\left(x\right) \longrightarrow \text{PDE}$$

Robin B.C.

$$\begin{cases} \alpha f + \beta_1 \dfrac{\partial f}{\partial n} = g_1(x) & \text{................ at } \Gamma_1 \\ \alpha f + \beta_2 \dfrac{\partial f}{\partial n} = g_2\left(x\right) & \text{.............. at } \Gamma_2 \end{cases}$$

Γ_1 ├──────┤ Γ_2

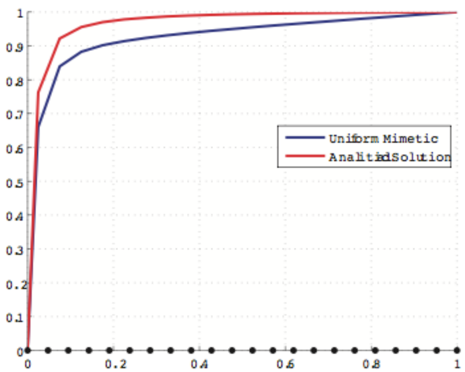
$x \in [0,1]$

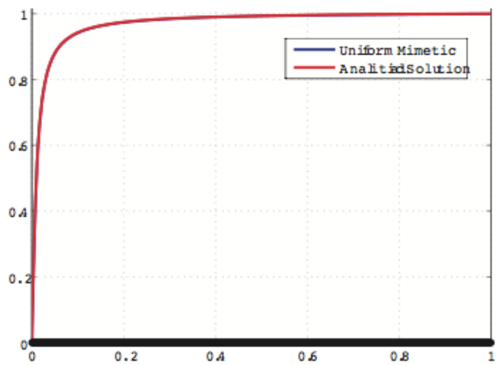
$$F\left(x\right) = \frac{2 \times 10^6}{\arctan(100)}\left[\frac{x}{(1 + 1 \times 10^4 x^2)^2}\right]$$

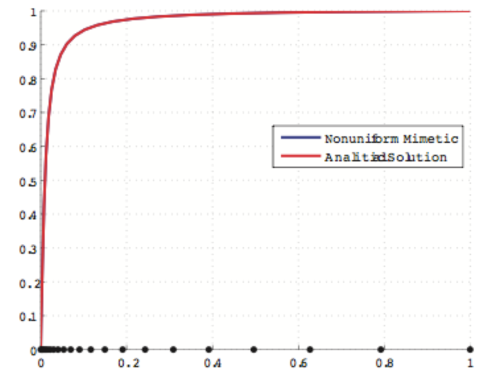
$\alpha = 1,$ $\beta_1 = \beta_2 = 1$ $C = \frac{100}{1+10^4},$ $g_1\left(x\right) = \frac{100}{\arctan\left(100\right)},$ $g_2\left(x\right) = 1 + \frac{100}{\arctan\left(100\right)\left(1+1\times 10^4\right)}$

Numerical Implementation -- 3

Uniform	Uniform	Adapted







Error Analysis

mesh	$\| \bullet \|_{\infty}$	$\| \bullet \|_{L_2}$
Uniform	$1.18 \times 10^{9} \times h^{4.17}$	$9.1 \times 10^{8} \times h^{4.17}$
Adapted	$11029.6 \times h^{3.98}$	$8105.84 \times h^{3.98}$

$$\| f - \mathrm{f} \|_{\infty} = \max\left\{ \left| f_{i+\frac{1}{2}} - \mathrm{f}_{i+\frac{1}{2}} \right|, \quad i = 0, \ldots, n-1 \right\}$$

$$\| f - \mathrm{f} \|_{L_2} = \sqrt{\sum_{i=0}^{n-1} \left(f_{i+\frac{1}{2}} - \mathrm{f}_{i+\frac{1}{2}} \right)^2 Volcell_{i+\frac{1}{2}}}$$

4 – Full Tensor Problem,
curvilinear boundary

Numerical Implementation -- 4

$$-div\left(K\ grad\left(f\left(\vec{x}\right)\right)\right) = F\left(\vec{x}\right) \longrightarrow \text{PDE}$$

$$-\mathrm{DKG}\ \ \mathrm{f} = \mathrm{F}$$

$$\begin{pmatrix} \mathrm{G}_x \\ \mathrm{G}_y \end{pmatrix}$$

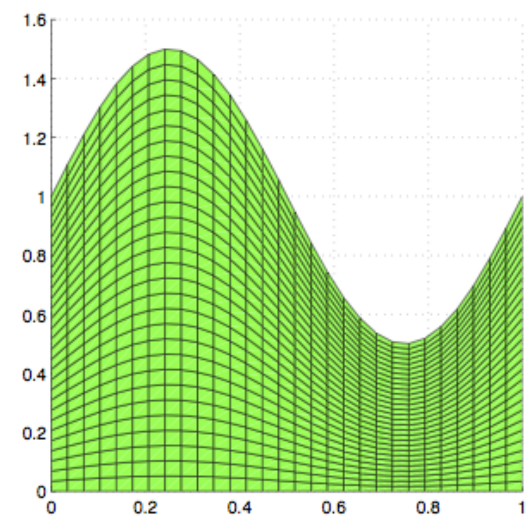
$$K = R\ D\ R^T$$

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

$$\theta = \frac{3\pi}{12}$$

$$d_1 = 1 + 2x^2 + y^2 + y^5$$

$$d_2 = 1 + x^2 + 2y^2 + x^3$$



Numerical Implementation -- 4

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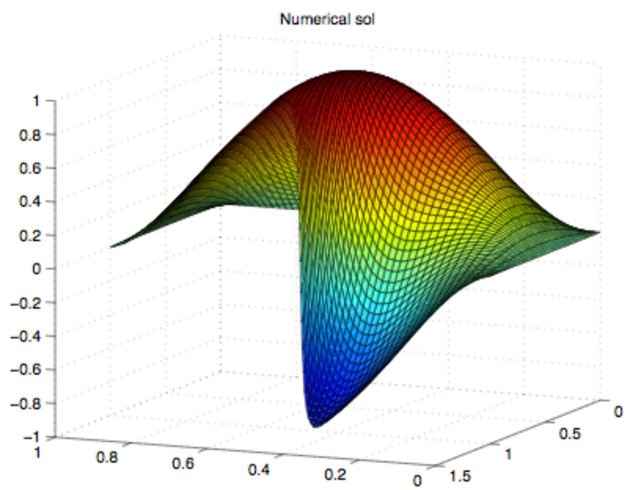

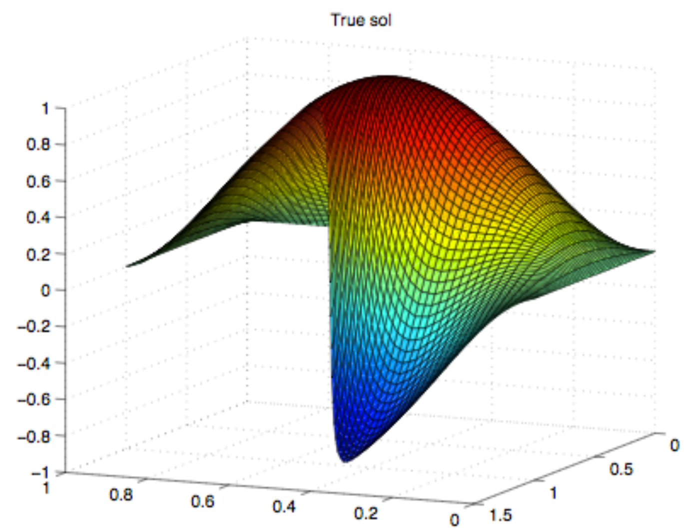

Error Analysis

h	Max Error	p
0.0625	0.025904	1.88
0.03125	0.007030	1.96
0.015625	0.001804	--

h	Norm-2	p
0.0625	0.004590	1.97
0.03125	0.001166	1.98
0.015625	0.000294	--

Conclusions

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