Acoustic Source Identification using Multiple Frequency Information

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Outline of the talk

- Introduction
  - Historical Remarks
  - Related Technologies
- Mathematical Modeling
  - Theoretical Results
  - Numerical Approach
- Numerical Testing

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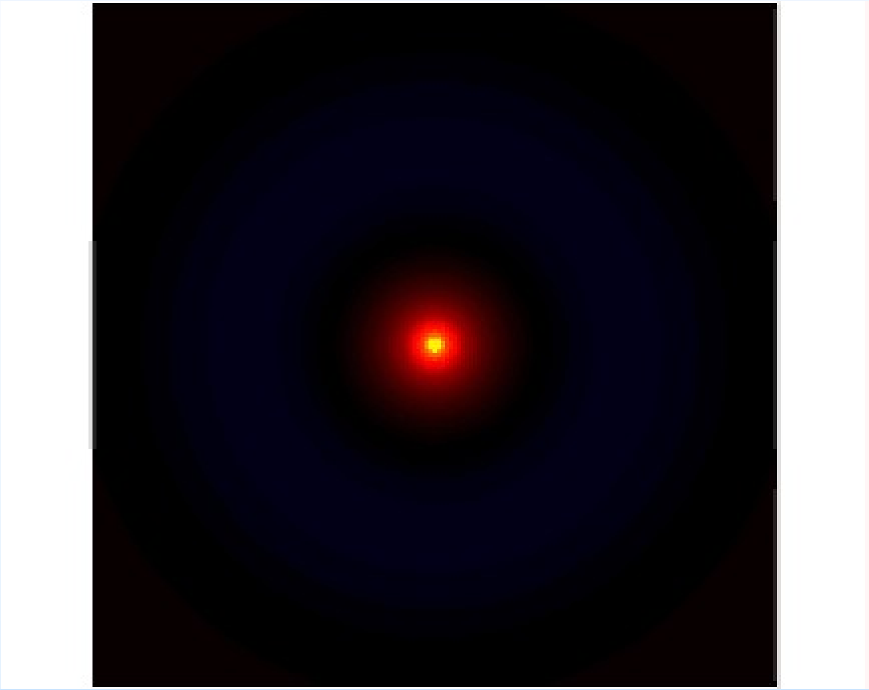
Introduction: Historical Remarks

Wave equation

$$\left(\Delta - \frac{1}{c^2}\frac{\partial}{\partial t}\right) u(t,x) = F(t,x)$$

$$t > 0, \quad x \in \mathbb{R}^d, d = 2, 3$$



- Noise Sources justify the formation of sound.
- Identification of location of sources allows the adequate reduction of noise.
- Important issue for the identification of enemy objects.

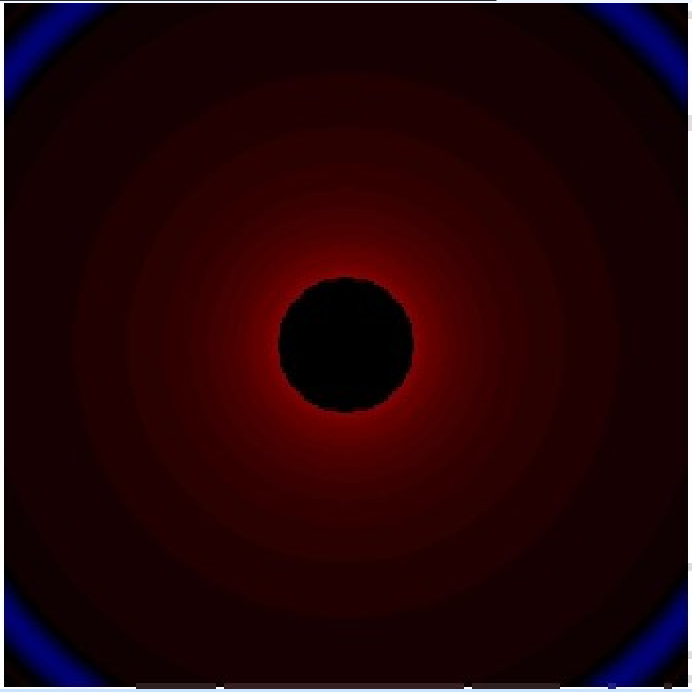
Introduction: Historical Remarks

Wave equation

$$\left(\Delta - \frac{1}{c^2}\frac{\partial}{\partial t}\right) u(t,x) = F(t,x)$$

$$t > 0, \quad x \in \mathbb{R}^d, d = 2, 3$$

$$F(t,x) = \delta(t) j_0(\pi x)$$



Novikov(1935), C. Muller (1955)

Non-uniqueness results

$$\left(\Delta + k^2\right) u(k,x) = F(k,x)$$

$$k > 0, \quad x \in \mathbb{R}^d, d = 2, 3$$

$$k = \frac{\omega}{c}$$

Porter (1969), Bojarski (1973)

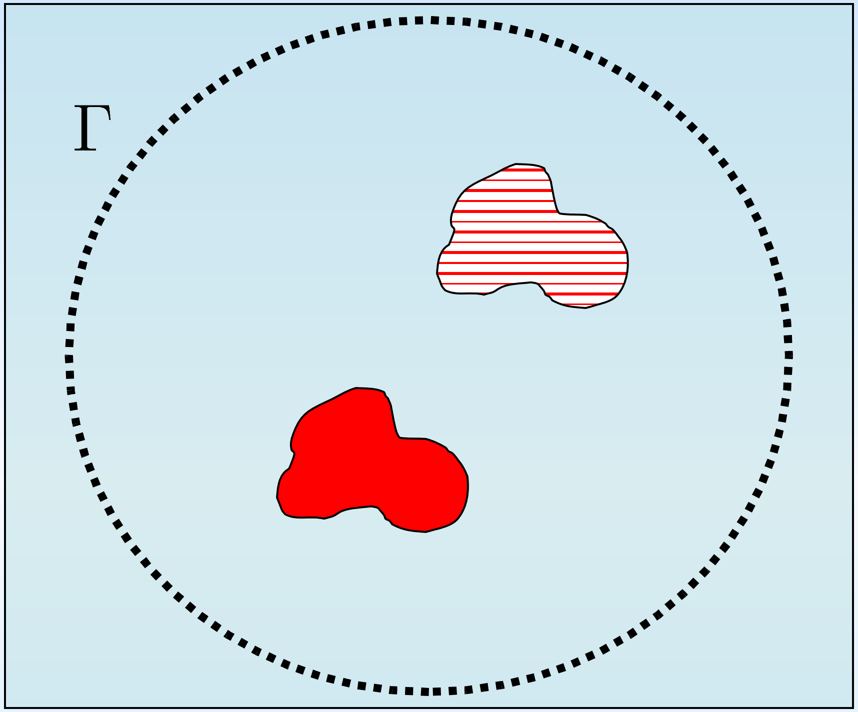
$$k = 2\pi, \quad u(2\pi, x) = 0, |x| > 1.$$

Introduction: Historical Remarks

Porter (1969), Bojarski (1973)

$$p(\mathbf{x},\omega) = \int_V F(\mathbf{y},\omega) j_0(\omega|\mathbf{x}-\mathbf{y}|/c_0) d\mathbf{x}$$

- Non-uniqueness allows the energy constraints in the design problems.
- Creation of decoys.



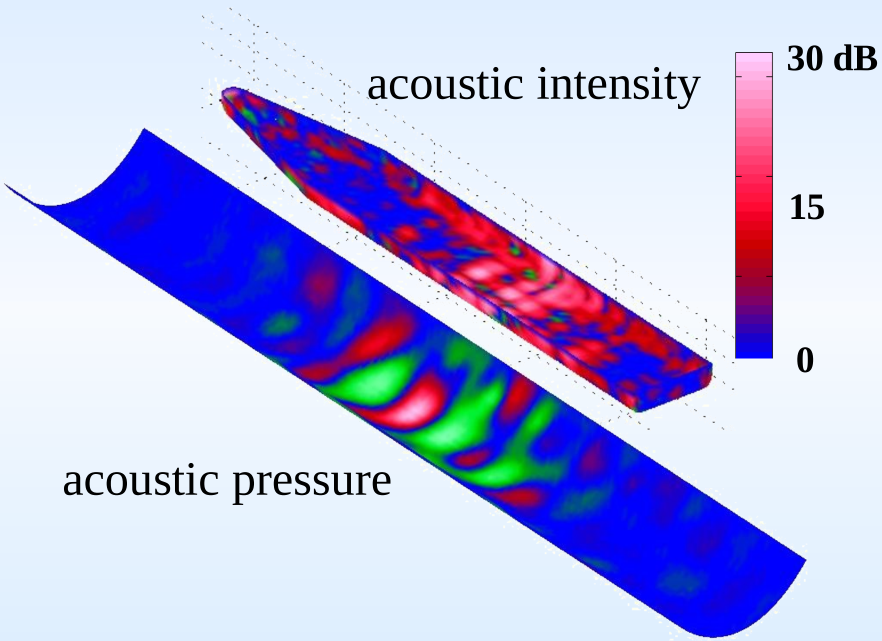
Bleistein & Cohen (1977), Devaney, Wolf , Lahaie, Marengo (80-90's)
Regularization methods, acoustic and electromagnetic cases.

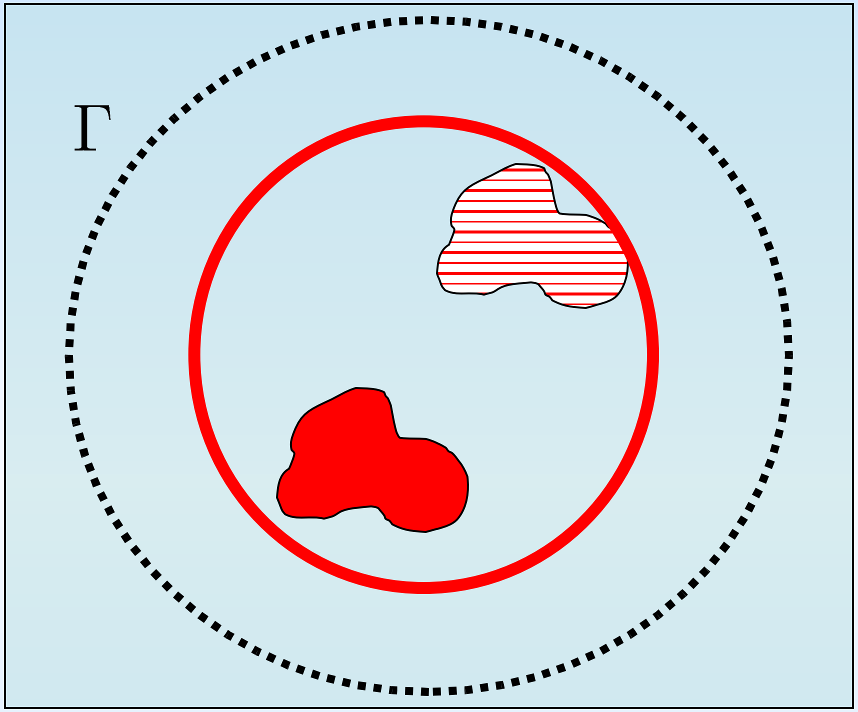
H. Moses (1984)
Utilization of time data. Uniqueness results. Volumetric data required.

Badia (1998), Ikehata (1999)
Uniqueness with Cauchy data.

Introduction: Related Problems

- **Nearfield Acoustic Holography (NAH) E.G. Williams et.al.**

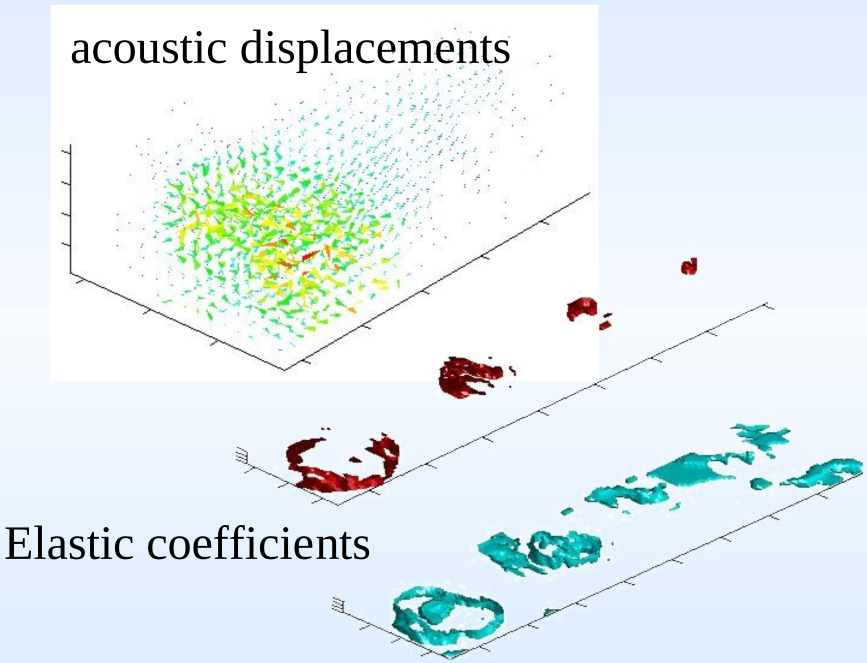


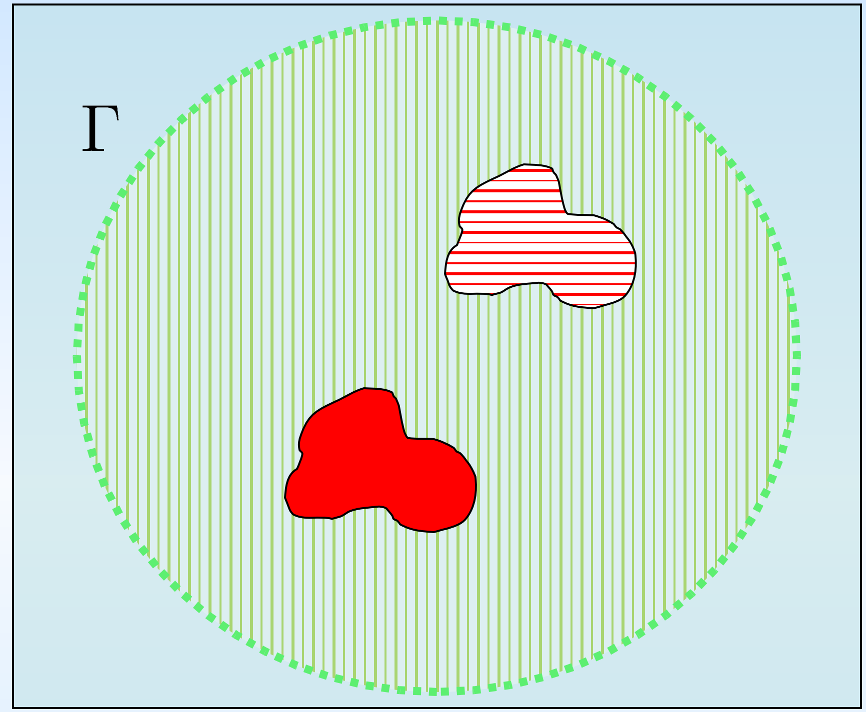


Reconstruction of the acoustic field in the surface of the structure from surface pressure measurements.

Introduction: Related Problems

- **Elastography**
  A.J. Romano et.al.





Reconstruction of elastic properties inside the structure from volumetric displacement measurements.

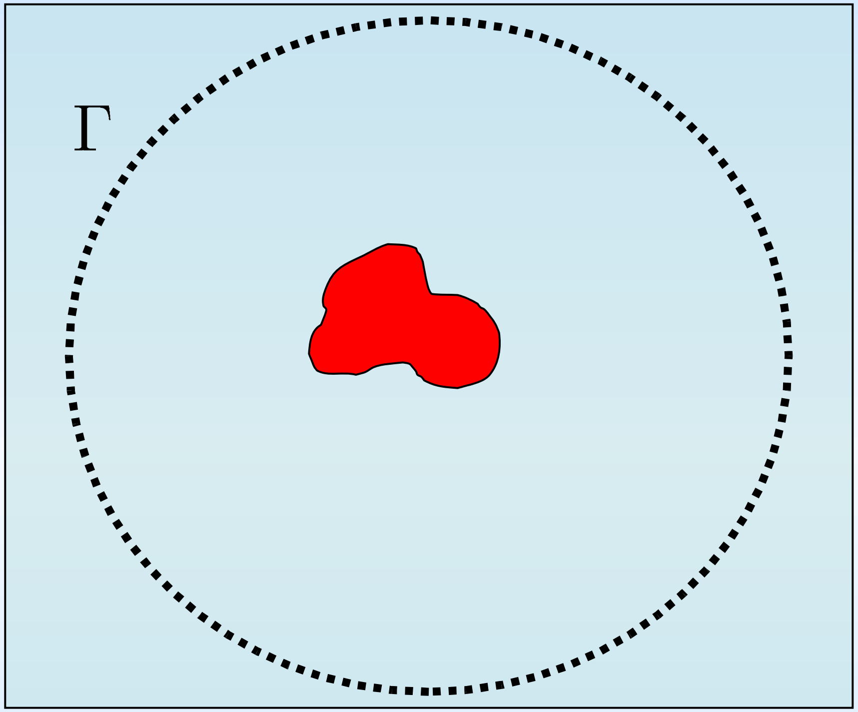
Mathematical Modeling

- **New Original Approach**

- Mathematical point of view: Uniqueness is given with the use of frequency information. The proof is constructive and can be generalized to many physical models.

- There is no solution of integral operators, only computation of surface integrals. Potentiality to do real-time reconstructions.

- Theoretical results support the explicit reconstruction when measurements in a part of the surface is known.



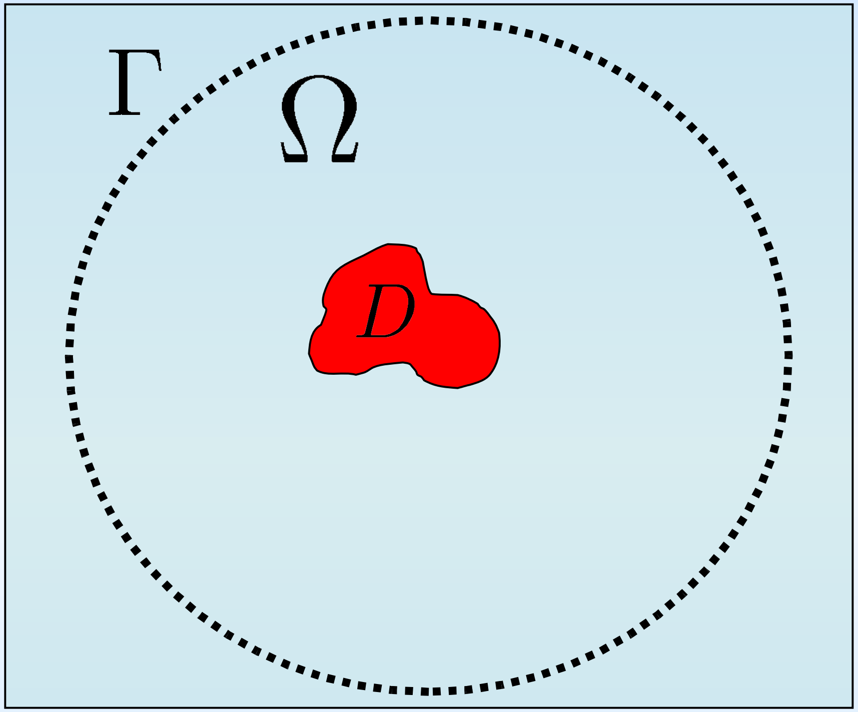
Mathematical Modeling

$$\left(\Delta + k^2\right) u(k,x) = F(k,x)$$

$$k > 0, \quad x \in \mathbb{R}^d, d = 2, 3$$

$$F(k,x) = g(k) f(x)$$

$$g > 0 \qquad \text{supp } f \subset D$$



Forward Problem

given $F(k,x) \in (0,K] \times \Omega$

recover $u(k,x) \in (0,K] \times \Omega$

Inverse Problem

given $u(k,x) \in (0,K] \times \Gamma$, $g > 0$

recover $f(x) \in \Omega$

Theoretical Results

Eller, Valdivia (2009), Uniqueness

Inverse Problem

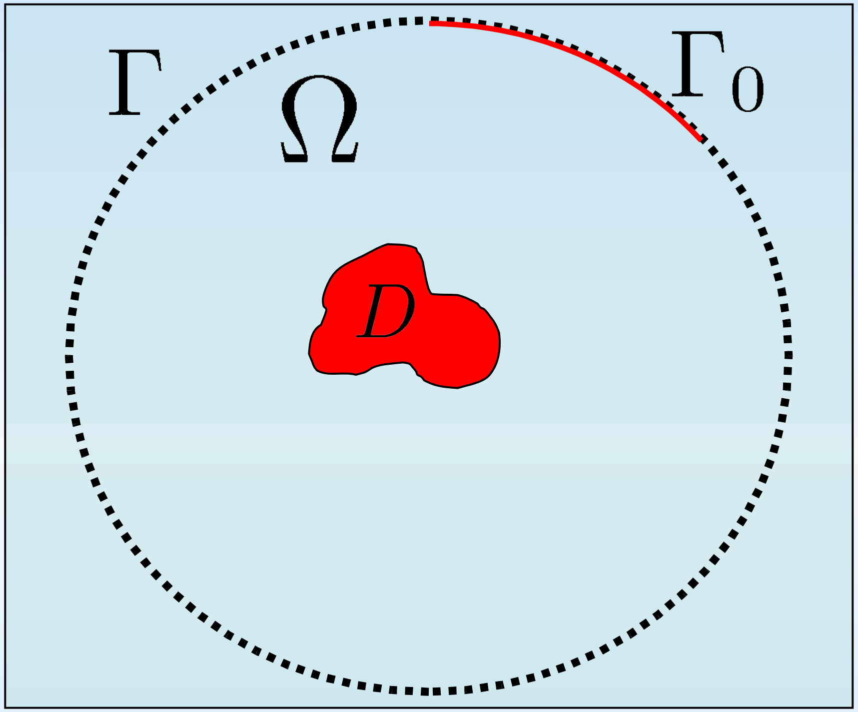
a) $u(k,x) \in (0,K] \times \Gamma$

b) $\partial_\nu u(k,x) \in (0,K] \times \Gamma$

c) $u(k,x), \partial_\nu u(k,x) \in (0,K] \times \Gamma_0$

Given a), b), or c) $g > 0$

recover $f(x) \in \Omega$



M.Eller and N.Valdivia, “Acoustic Source Identification using Multiple Frequency Information” *Inverse Problems* 25, 2009

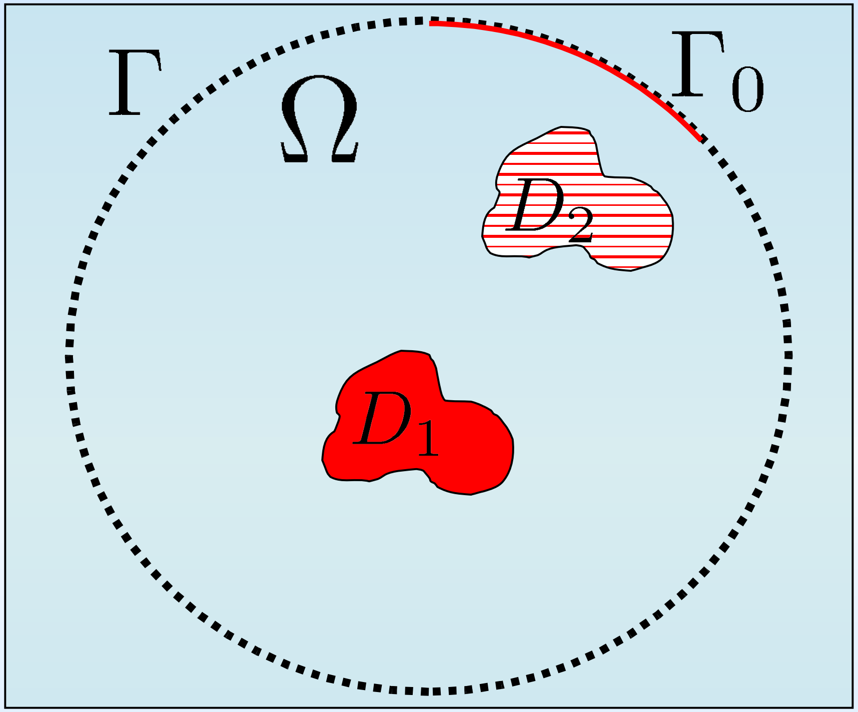
Theoretical Results

Proof

$$\left(\Delta + k^2\right) u_i(k,x) = F_i(k,x), i = 1,2$$
$$k > 0, \quad x \in \mathbb{R}^d, d = 2,3$$

$$F_i(k,x) = g(k) f_i(x)$$
$$g > 0 \quad \text{supp } f_i \subset D_i$$

a) $u_1(k,x) = u_2(k,x) \in (0,K] \times \Gamma$
$g > 0$

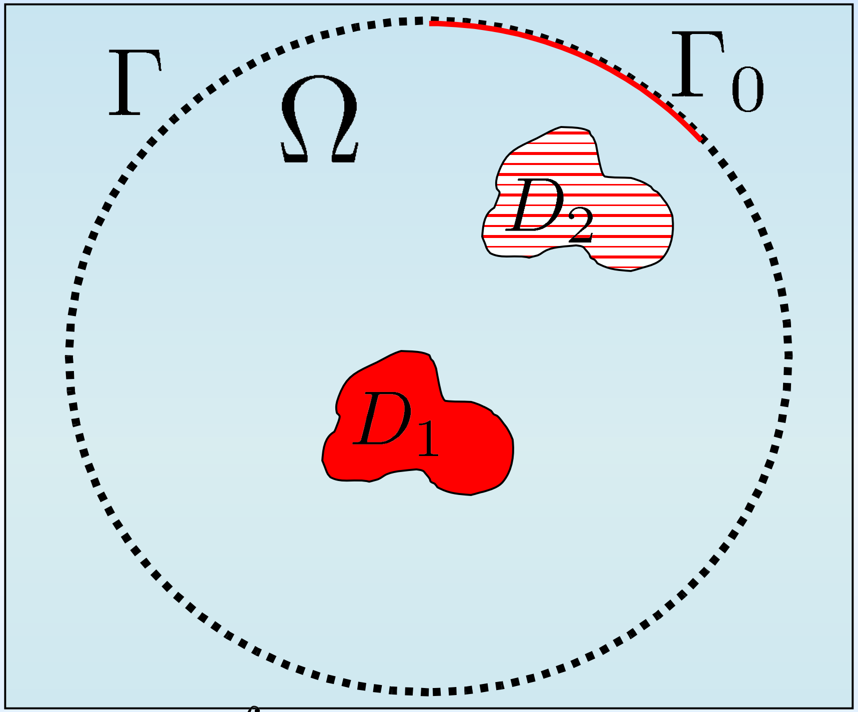


Theoretical Results

Lemma

$$\int_{D_1} f_1 w_k dx = \int_{D_2} f_2 w_k dx$$

$$\Delta w_k + k^2 w_k = 0, \text{ in } \Omega$$



Proof

$$g(k)\int_{D_1} f_1 w_k dx = \int_{\Omega} g(k) f_1 w_k dx$$

$$= \int_{\Omega} (\Delta u_1 + k^2 u_1) w_k dx = \int_{\Gamma} \{\partial_\nu u_1 w_k - u_1 \partial_\nu w_k\} d\Gamma.$$

$$g(k)\int_{D_2} f_2 w_k dx = \int_{\Gamma} \{\partial_\nu u_2 w_k - u_2 \partial_\nu w_k\} d\Gamma.$$

Theoretical Results

Proof

assume $D_1 \neq D_2$

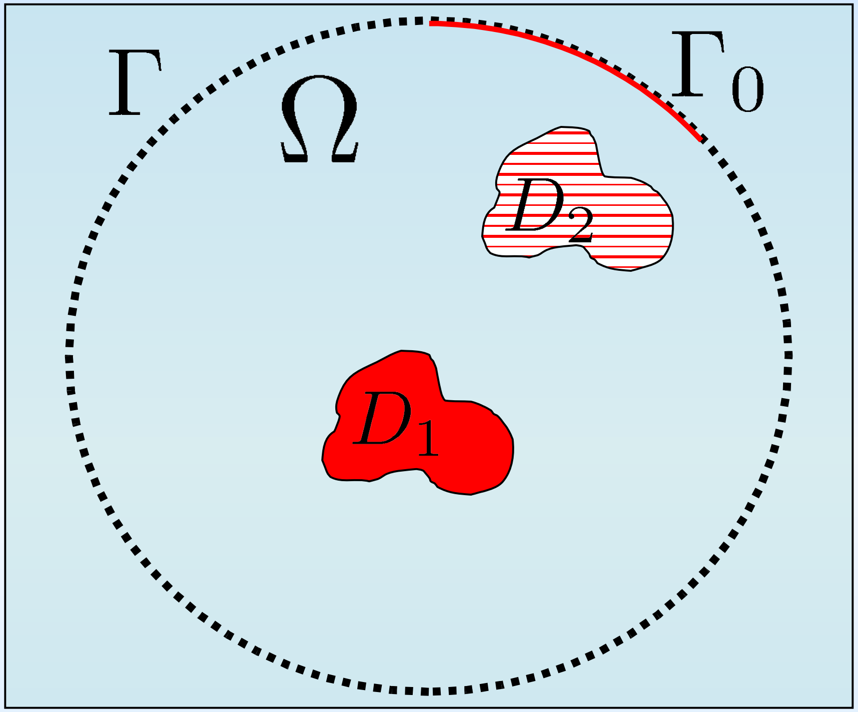
$$\int_{D_1} f_1 w_k dx = \int_{D_2} f_2 w_k dx$$

$$f = f_1 - f_2$$

$$\int_{\Omega} f w_k dx = 0$$

$$-\Delta w_n = \lambda_n w_n, \text{ in } \Omega$$
$$w_n = 0, \text{ on } \Gamma$$

$$\int_{\Omega} f w_n dx = 0, \quad n = 1, 2, \ldots$$



$$f_1 = f_2$$
$$D_1 = D_2$$

Numerical Approach

Explicit representation of solution

$$f = \sum_n A_n w_n$$

Properties of Eigen-functions

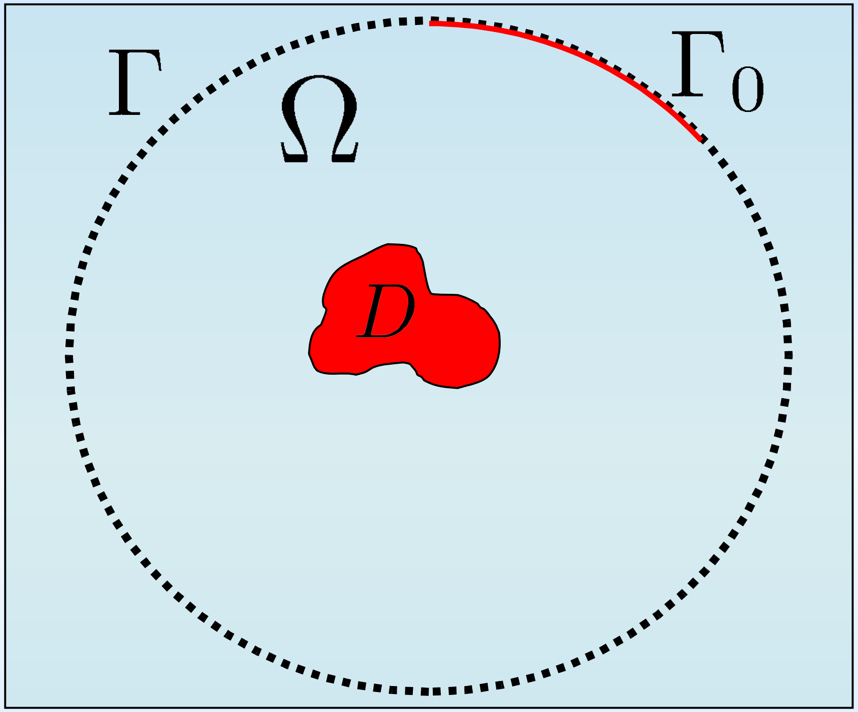
$$A_n = \int_\Gamma \left\{ u(k_n, .)\partial_\nu w_n - \partial_\nu u(k_n, .) w_n \right\} dS$$

Neumann Eigen-functions

$$A_n = -\int_\Gamma \partial_\nu u(k_n, .) w_n dS$$

Dirichlet eigenfunctions

$$A_n = \int_\Gamma u(k_n, .)\partial_\nu w_n dS$$



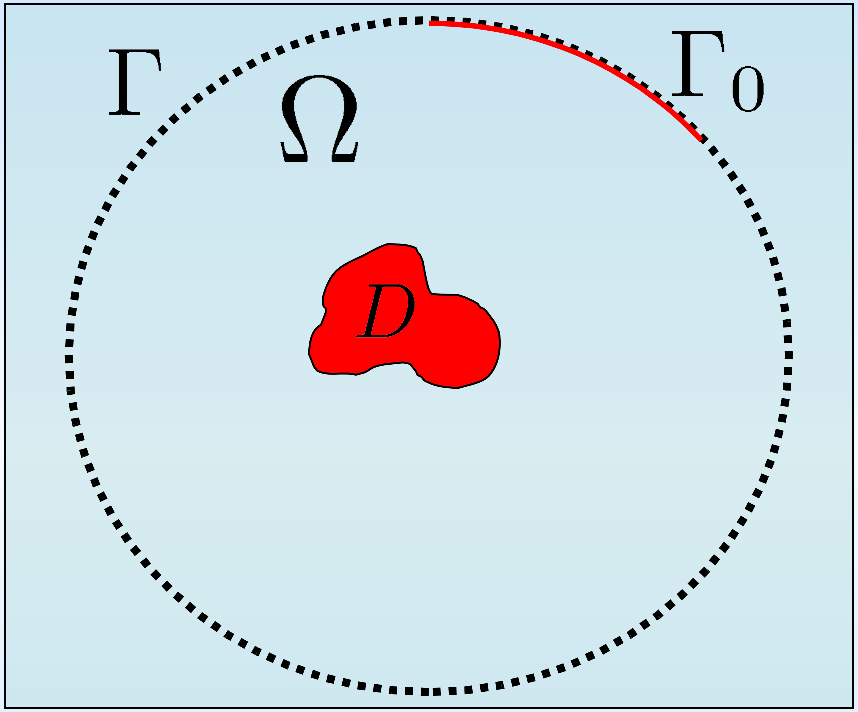
Numerical Approach

Explicit representation of solution

$$f = \sum_n A_n w_n$$

Error estimate (Stability)

$$\left\| f - \sum_{n=1}^{N} A_n w_n \right\| \leq C \|f\|_s N^{1/2 - s/d}$$



$$g > 0 \quad f(x) \in H_0^s(\Omega) \quad s > \frac{d}{2}$$

$$k > 0, \quad x \in \mathbb{R}^d, d = 2, 3$$

Numerical Testing

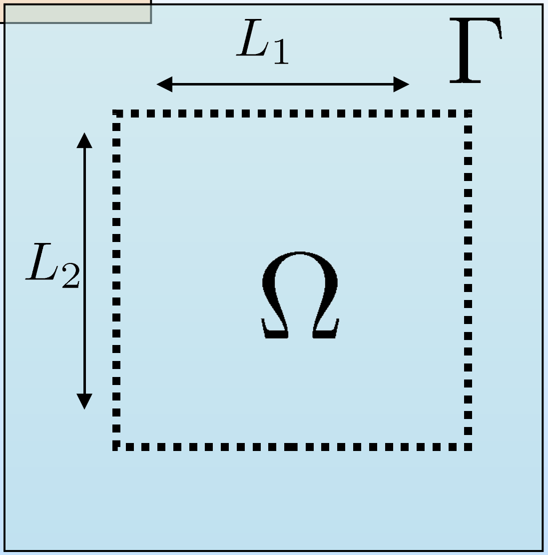
Eigenfunctions basis

$$Q_{lmn}(x) = \frac{\sqrt{8}}{\sqrt{L_1 L_2 L_3}} \sin\left(\frac{l\pi x_1}{L_1}\right) \sin\left(\frac{m\pi x_2}{L_2}\right) \sin\left(\frac{n\pi x_3}{L_3}\right)$$

$$k_{lmn} = \sqrt{\left(\frac{l\pi}{L_1}\right)^2 + \left(\frac{m\pi}{L_2}\right)^2 + \left(\frac{n\pi}{L_3}\right)^2}$$

$$Q_{mn}(x) = \frac{2}{\sqrt{L_1 L_2}} \sin\left(\frac{m\pi x_1}{L_1}\right) \sin\left(\frac{n\pi x_2}{L_2}\right)$$

$$k_{mn} = \sqrt{\left(\frac{m\pi}{L_1}\right)^2 + \left(\frac{n\pi}{L_2}\right)^2}$$



Numerical Testing

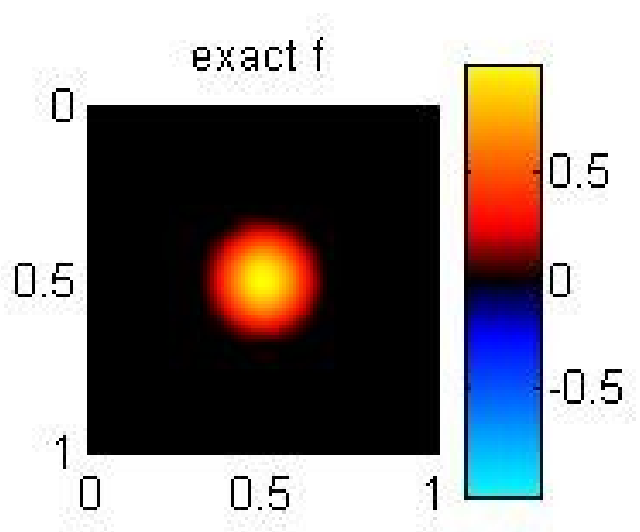
2D Data

given $f(x) = \begin{cases} 0.5\cos(5\pi|x-d|) + 1/2, & |x-d| < 1/5 \\ 0, & else \end{cases}$

Compute

$$u(k,x) = g(k)\int_{\Omega} \Phi(x,y) f(y) dy$$

$$\Phi(x,y) = \frac{i}{4} H_0(k|x-y|)$$



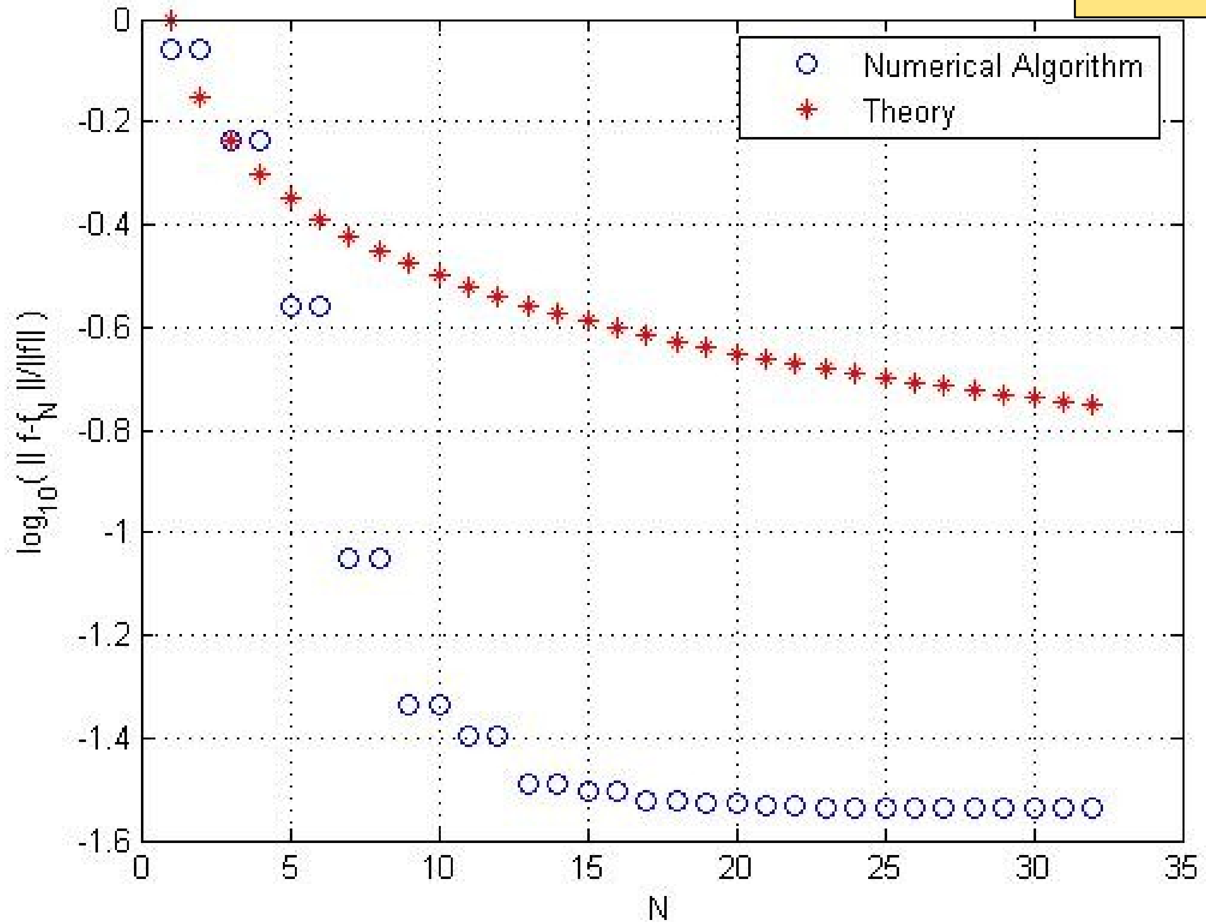
Dirichlet eigenfunctions

$$A_n = \int_{\Gamma} u(k_{mn}, .)\partial_{\nu} Q_{mn} dS$$

Numerical Testing

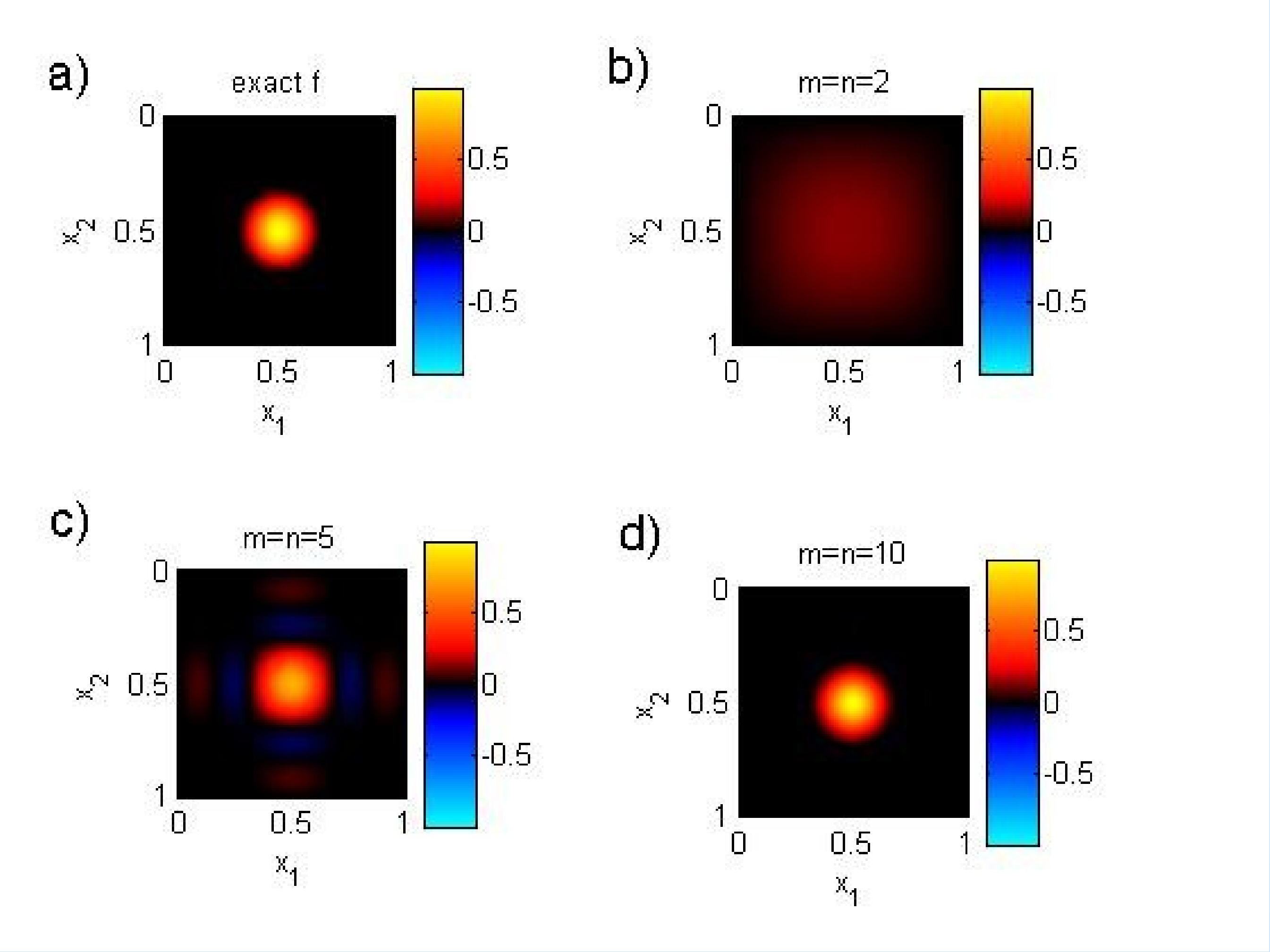
Error estimate (Stability)

$$\left\| f - \sum_{n=1}^{N} A_n w_n \right\| \le C \|f\|_s N^{-1/2}$$



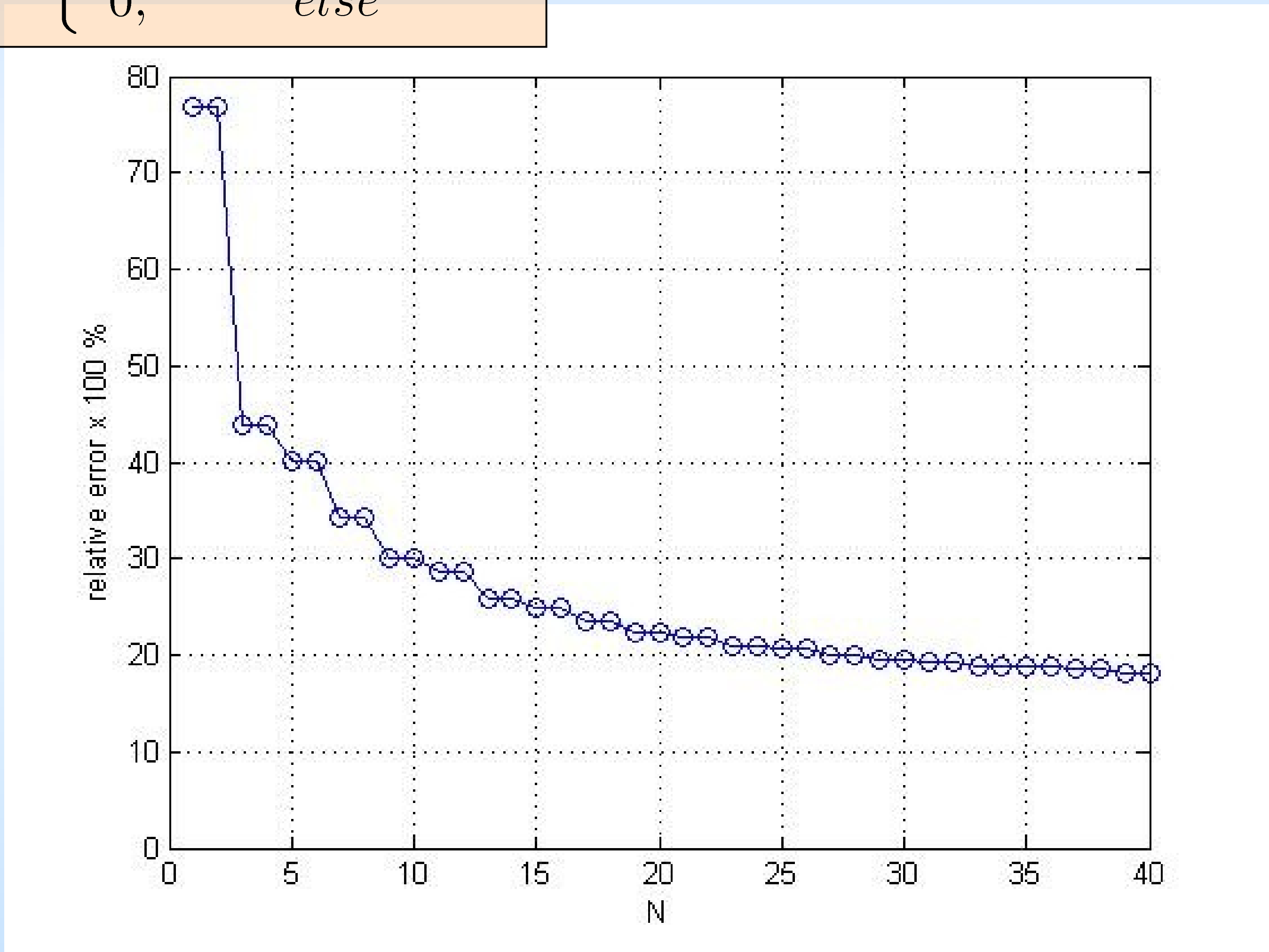
$$f(x) \in H_0^2(\Omega)$$

Numerical Testing

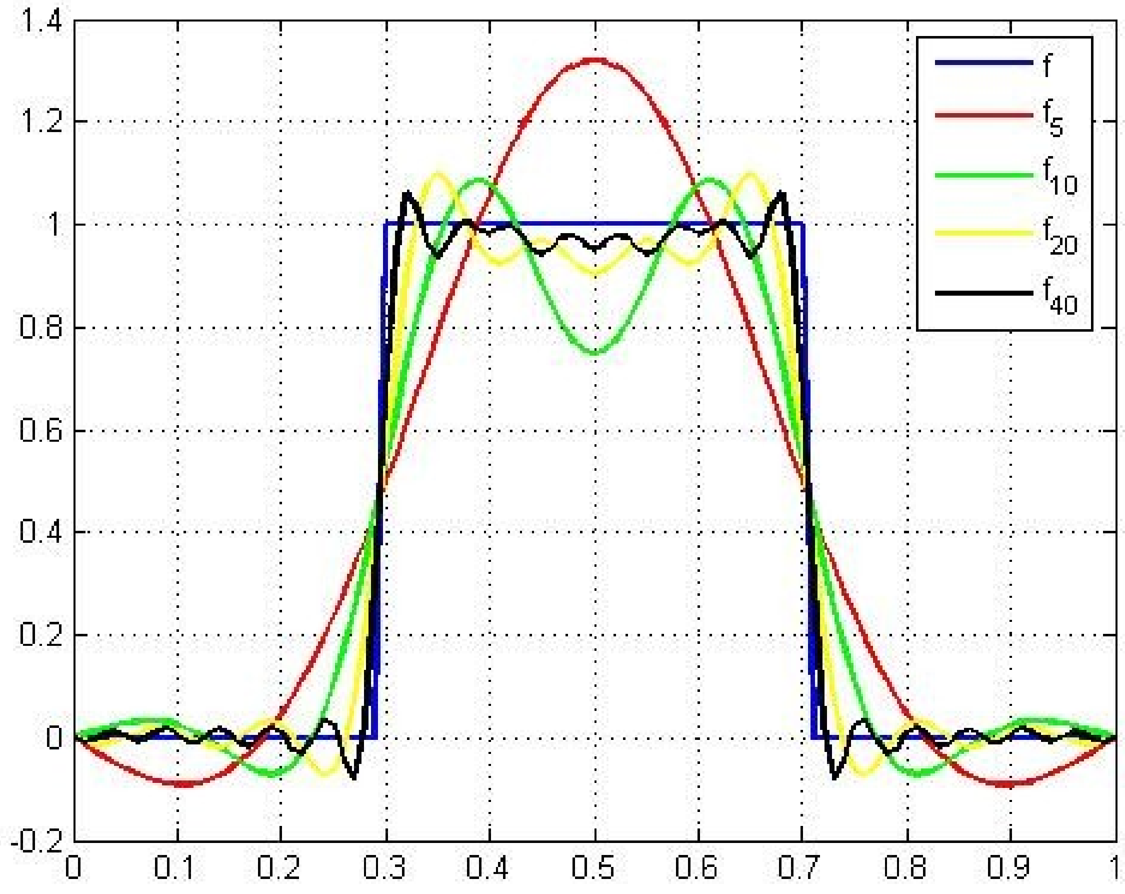


Numerical Testing

given $f(x) = \begin{cases} 1, & |x-d| < 0.5 \\ 0, & else \end{cases}$

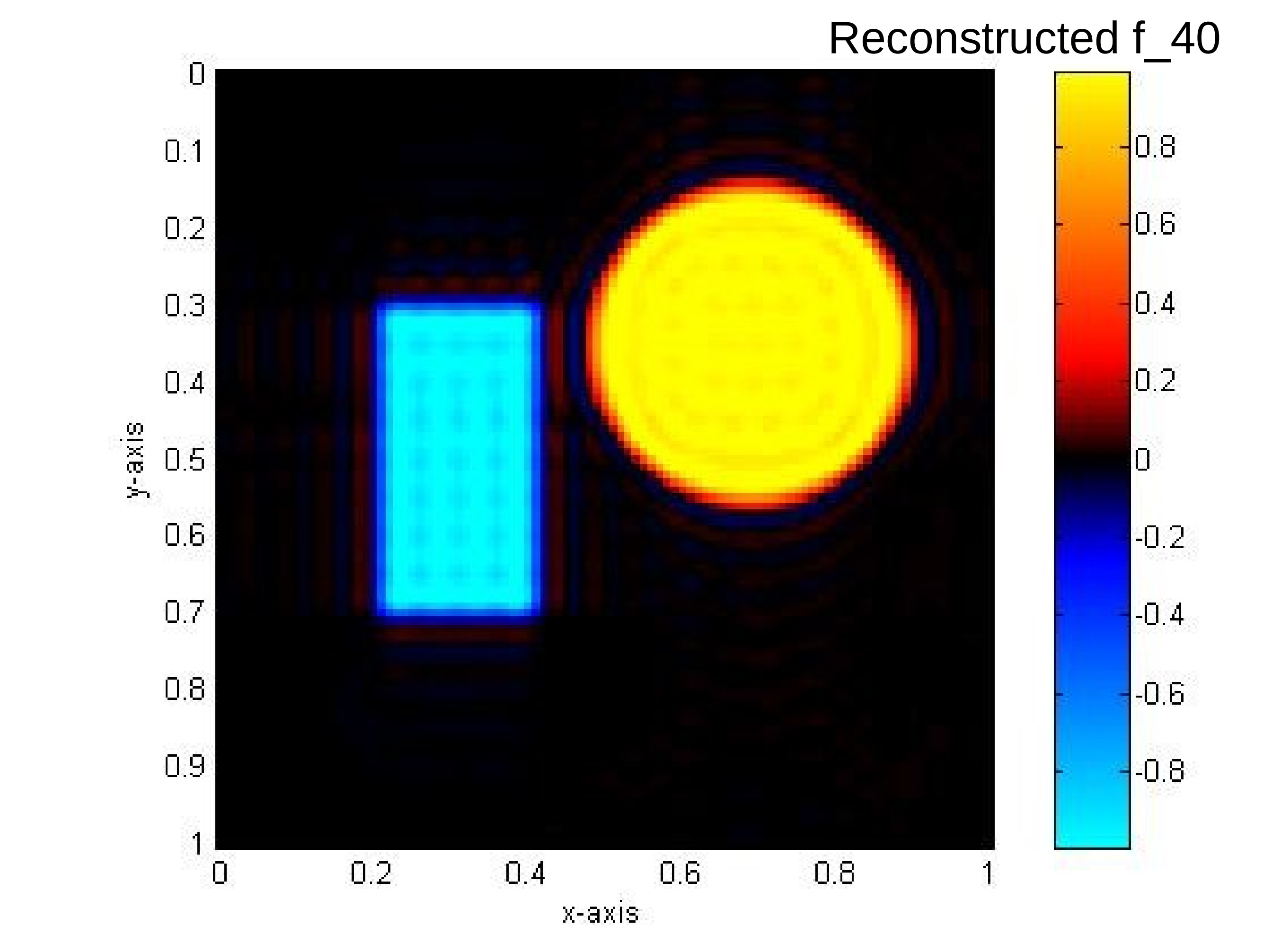


Numerical Testing



Numerical Testing

given $f(x) = \chi_B - \chi_Q$



Numerical Testing

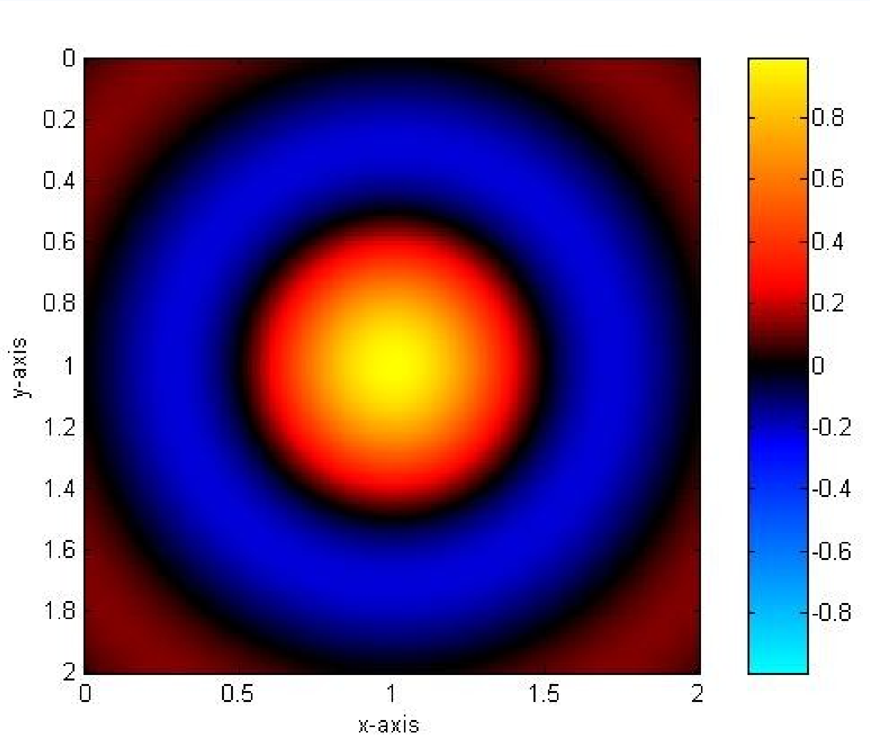
3D Data

given $f(x) = j_0(2\pi|x|)$

Compute $u(k,x) = ikh_0(k|x|)\int_0^1 j_0(2\pi r)j_0(kr)r^2 dr$

Dirichlet eigenfunctions

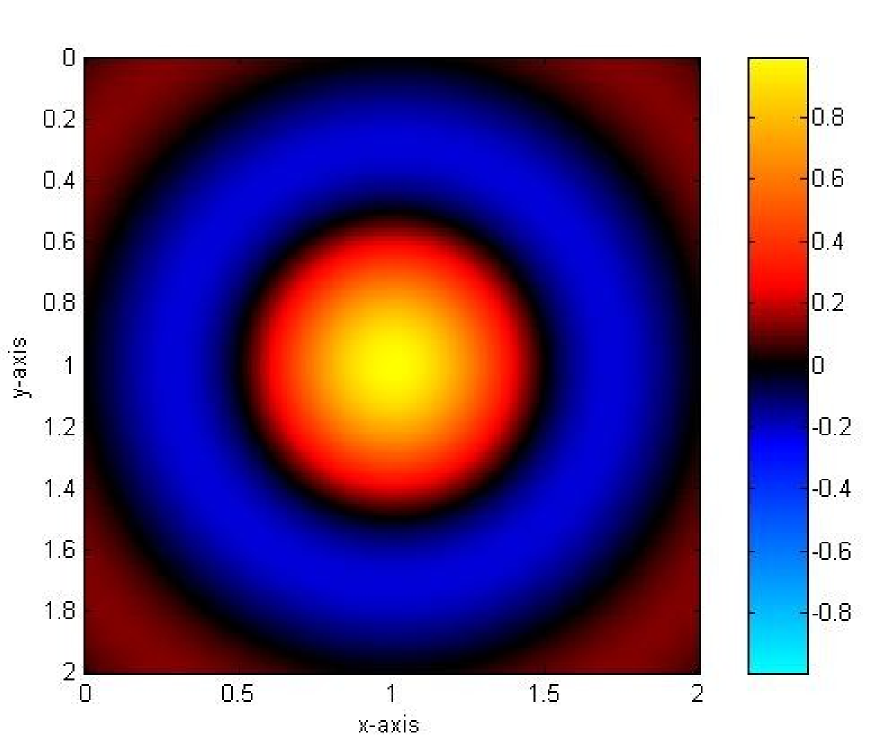
$$A_{lmn} = \int_\Gamma u(k_{lmn},.)\partial_\nu Q_{mn} dS$$



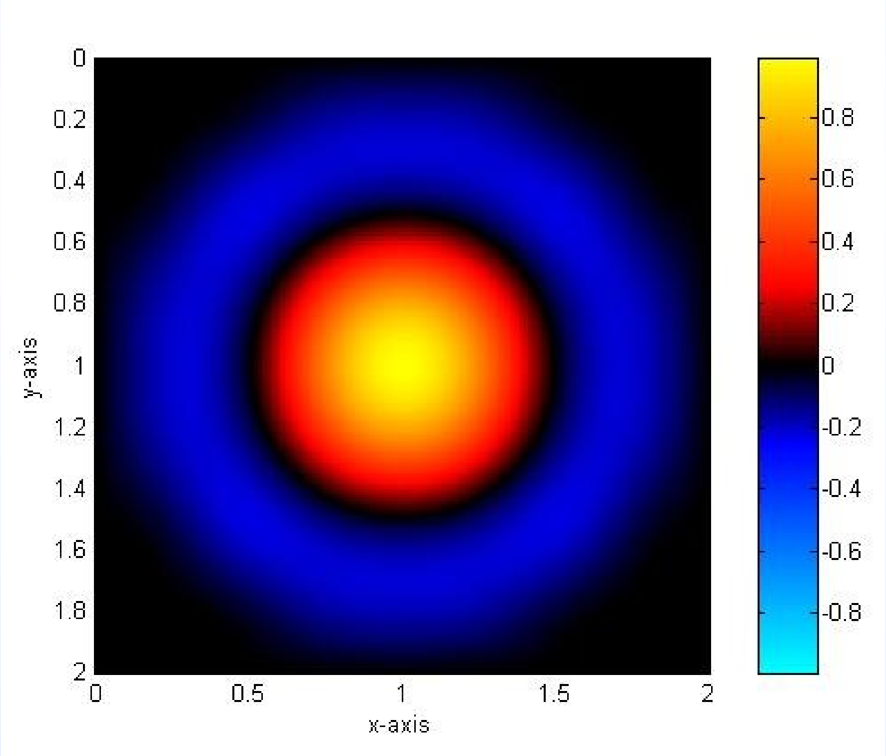
Numerical Testing

3D Data

$$f(x) = j_0(2\pi|x|)$$



Reconstructed f_8



Conclusions

- The source function is expressed explicitly in a formula that depends on the geometry.
- The method is general and can be extended to other models.
- Theory allows us to believe that reconstructions can be obtained from measurements in parts of the surface.
- Real time calculations can be performed.