High Order Mimetic Differential Operators

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Abstract

- *Mimetic Operators satisfy a discrete analog of the divergence theorem and they are used to create/design conservative/reliable numerical representations to continuous models.*
- *We will present a methodology to construct mimetic versions of the divergence and gradient operators which exhibit high order of accuracy at the grid interior as well as at the boundaries.*
- *Mimetic conditions on discrete operators are stated using matrix analysis and the overall high order of accuracy determines the bandwidth parameter.*
- *Test cases:*
  - *2-D elliptic equations with full tensor coefficients*
  - *Elastic wave propagation under free surface and shear rupture boundary conditions*

Outline

- Mimetic discretization
  - Principles/Foundations
  - Methodology of construction of mimetic operators
  - Second order and Higher order results
- Applications to elliptic problems and hyperbolic equations

Mimetic Discretization

- First order invariant continuum operators
  - div, grad, curl

- Discrete operators
  - DIV, GRAD, CURL
  - Preserve (exactly) many of the fundamental properties of the continuum differential operators (important to preserve conservation laws)

Goal

- Construct local high-order DIV, GRAD, CURL on uniform and non-uniform grids that mimic properties of continuum differential operators div, grad, curl

The Problem

- Find higher-order approximations GRAD, DIV to grad (∇) and div ($\nabla \cdot$)
- Divergence Theorem

$$\int_{\Omega} \nabla \cdot \vec{v} f \, dV + \int_{\Omega} \vec{v} \nabla f \, dV = \int_{\partial \Omega} f \vec{v} \, \vec{n} \, dS$$

1-D Mimetic Operators

Castillo & Grone (2003)

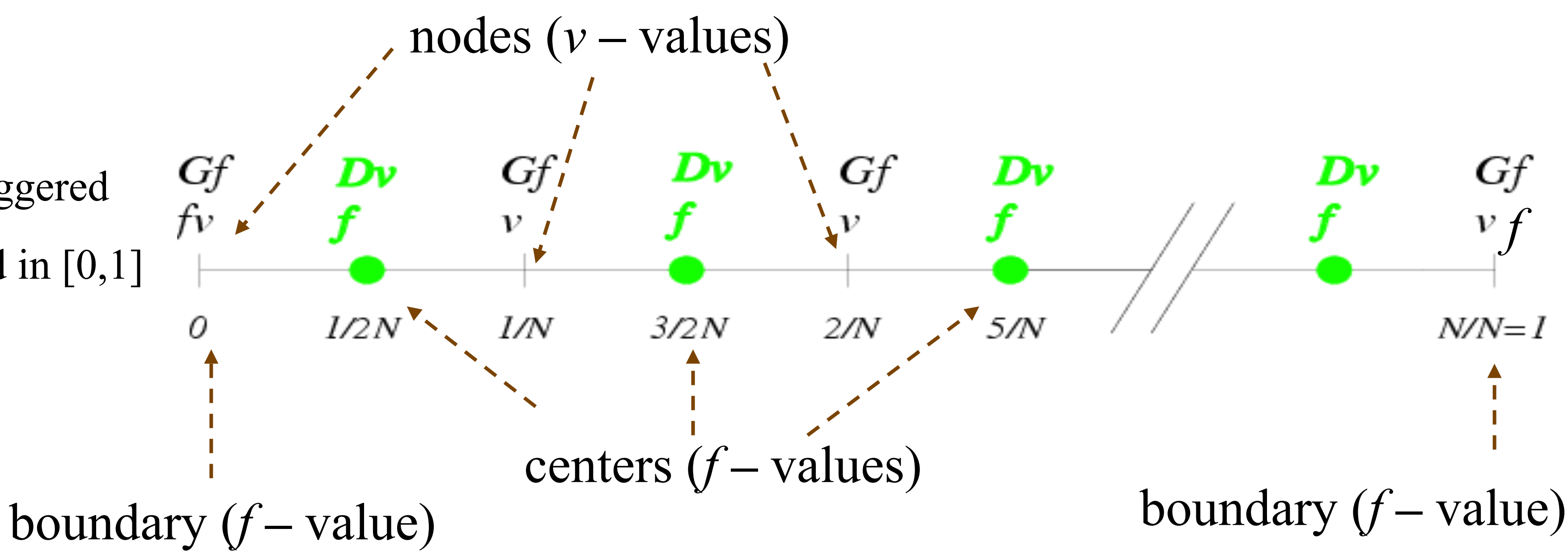
Discrete

DIV theorem

$$\langle \mathbf{D}v, f\rangle_Q + \langle v, \mathbf{G}f\rangle_P = \langle \mathbf{B}v, f\rangle \qquad (v, f \text{ are vectors})$$

Staggered

grid in [0,1]



Higher-Order Case: Matrix Formulation

- Function v becomes an N-tuple
- Divergence operator, N by (N+1) matrix $\mathbf{D}$
- If $e = \begin{pmatrix} 1, & 1, & \cdots & 1 \end{pmatrix}^{\mathrm{T}}$ then $\mathbf{D}e = 0$ can be expressed as

$$\mathbf{D}e = 0$$

- Discrete analog of global conservation condition

$$\left\langle \mathbf{D}v, e \right\rangle = v_N - v_0$$

- $\mathbf{D}$ has column sums equals –1, 0, …, 0, 1,

$$e^{\mathrm{T}}\mathbf{D} = \begin{pmatrix} -1, & 0, & \cdots & 0, & 1 \end{pmatrix}^{\mathrm{T}}$$

Boundary Symmetry

- Permutation matrix

$$P_k = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

- $\mathbf{D}$ should satisfy: $P_n \mathbf{D} P_{n+1} = -\mathbf{D}$ (Centro-skew-symmetric)

Summary of Matrix Conditions

- Consistency
- Mimetic/Column sums
- Local
- Interior symmetry
- Boundary symmetry

Fourth-Order Divergence

- Find A s.t. $H = (A)$ is a uniformly 4th-order DIV operator

$$H = H(A) = \begin{bmatrix} A & & & & & & & 0 & 0 & 0 & 0 \\ & & & & & \ddots & & & & & \\ 0 & 0 & 0 & \frac{1}{24} & -\frac{9}{8} & \frac{9}{8} & -\frac{1}{24} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{24} & -\frac{9}{8} & \frac{9}{8} & -\frac{1}{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{24} & -\frac{9}{8} & \frac{9}{8} & -\frac{1}{24} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{24} & -\frac{9}{8} & \frac{9}{8} & -\frac{1}{24} & 0 \\ & & & & & & & & \ddots & & \\ 0 & 0 & 0 & 0 & & & & & & & A' \end{bmatrix}$$

Conditions Near The Boundary

- $\mathbf{a}$ satisfies
  - Row sum
  - Column sum
  - Order constraints

$$V_1 = V\left(1, \quad -1, \quad -3, \quad -5, \quad -7, \quad -9\right)$$

- Where

$$V\left(1, \quad -1, \quad -3, \quad -5, \quad -7, \quad -9\right) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -3 & -5 & -7 & -9 \\ 1 & 1 & 9 & 25 & 49 & 81 \\ 1 & -1 & 27 & -125 & -343 & -729 \\ 1 & 1 & 81 & 625 & 2401 & 6561 \end{bmatrix}$$

Conditions Near The Boundary (Cont'd)

- Similarly

$$V_2 = V\left(3, \quad 1, \quad -1, \quad -3, \quad -5, \quad -7\right)$$
$$V_3 = V\left(5, \quad 3, \quad 1, \quad -1, \quad -3, \quad -5\right)$$
$$V_4 = V\left(7, \quad 5, \quad 3, \quad 1, \quad -1, \quad -3\right)$$

Matrix Form of The Conditions

$$M = \begin{bmatrix} V_1 & 0 & 0 & 0 \\ 0 & V_2 & 0 & 0 \\ 0 & 0 & V_3 & 0 \\ 0 & 0 & 0 & V_4 \\ I_6 & I_6 & I_6 & I_6 \end{bmatrix}$$

$$b^{\mathrm{T}} = \left(0, \quad -2, \quad 0, \quad 0, \quad 0, \quad 0, \quad -2, \quad 0, \quad 0, \quad 0, \quad \cdots \quad -1, \quad 0, \quad 0, \quad -\tfrac{1}{24} \quad \tfrac{13}{12} \quad -\tfrac{1}{24}\right)$$

$$M\mathbf{a} = b$$

Weighted Inner Product

- Find Q and D such that

$$\langle QDv, 1 \rangle = v_N - v_0$$

- Or

$$e^{\mathrm{T}} QD = \begin{pmatrix} -1, & 0, & \cdots & 0, & 1 \end{pmatrix}^{\mathrm{T}}$$

New Formulation

$$\hat{V}\left(1, \quad -1, \quad -3, \quad -5, \quad -7, \quad -9\right) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 9 & 25 & 49 & 81 \\ 1 & -1 & 27 & -125 & -343 & -729 \\ 1 & 1 & 81 & 625 & 2401 & 6561 \end{bmatrix}$$

New Formulation (Cont'd)

$$\hat{M} = \begin{bmatrix} \hat{V}_1 & 0 & 0 & 0 \\ 0 & \hat{V}_2 & 0 & 0 \\ 0 & 0 & \hat{V}_3 & 0 \\ 0 & 0 & 0 & \hat{V}_4 \\ I_6 & I_6 & I_6 & I_6 \end{bmatrix}$$

$$\hat{M}\mathbf{a} = \hat{b}$$

- 3 parametric-(α, β, γ) family of solutions

Fourth Order Operator D

$$D = \frac{1}{h}\begin{bmatrix} -\frac{4751}{5192} & \frac{909}{1298} & \frac{6091}{15576} & -\frac{1165}{5192} & \frac{129}{2596} & -\frac{25}{15576} & \cdots \\ \frac{1}{24} & -\frac{27}{24} & \frac{27}{24} & -\frac{1}{24} & 0 & 0 & \cdots \\ 0 & \frac{1}{24} & -\frac{27}{24} & \frac{27}{24} & -\frac{1}{24} & 0 & \cdots \end{bmatrix}$$

- Where the weights are

$$Q_0 = \frac{649}{576}, \quad Q_1 = \frac{143}{192}, \quad Q_2 = \frac{75}{64}, \quad Q_3 = \frac{551}{576}, \quad Q_4 = 1, \quad Q_5 = 1, \quad \cdots$$

Fourth Order Operator G

$$G = \frac{1}{h}\begin{bmatrix} -\frac{1152}{407} & \frac{10063}{3256} & \frac{2483}{9768} & -\frac{3309}{3256} & \frac{2099}{3256} & -\frac{697}{4884} & \cdots \\ 0 & -\frac{11}{12} & \frac{17}{24} & \frac{3}{8} & -\frac{5}{24} & \frac{1}{24} & \cdots \\ 0 & \frac{1}{24} & -\frac{27}{24} & \frac{27}{24} & -\frac{1}{24} & 0 & \cdots \end{bmatrix}$$

- Where the weights are

$$P_0 = \frac{407}{1152}, \quad P_1 = \frac{473}{384}, \quad P_2 = \frac{343}{384}, \quad P_3 = \frac{1177}{1152}, \quad P_4 = 1, \quad \cdots$$

First Results

- Grids
  - Staggered
  - Nodal
- Weighted inner products
  - Diagonal matrix
- High order gradients
- High order divergence
- Higher dimensions/Tensor product grids

2-D Elliptic Problems

(Huy Vu)

$$-div\left(K\, grad\, u\right) = F(x), \quad x \in \Omega$$

- K is a tensor function, $F\left(x\right)$ is a source term
- Robin boundary conditions

$$\beta\left(\hat{n}, K\, grad\, u\right) + \alpha\, u = \gamma, \quad x \in \partial\Omega$$

- α, β, γ are function given on $\partial\Omega$

Second Order Operators: D

- Simplest case: Discrete divergence ($\mathbf{D}$) defined by

$$(\mathbf{D}v)_{i+\frac{1}{2}} = \frac{v_{i+1} - v_i}{h}, \quad 1 \le i \le N-1$$

$$\mathbf{D} = \frac{1}{h}\begin{bmatrix} -1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{bmatrix} \in R^{N\times(N+1)}$$

Second Order Operators: G

$$G = \frac{1}{h}\begin{bmatrix} -\frac{8}{3} & 3 & -\frac{1}{3} & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & -1 & 1 & 0 \\ 0 & \cdots & 0 & \frac{1}{3} & -3 & \frac{8}{3} \end{bmatrix} \in R^{(N+1)\times(N+2)}$$

Second Order Operators: B

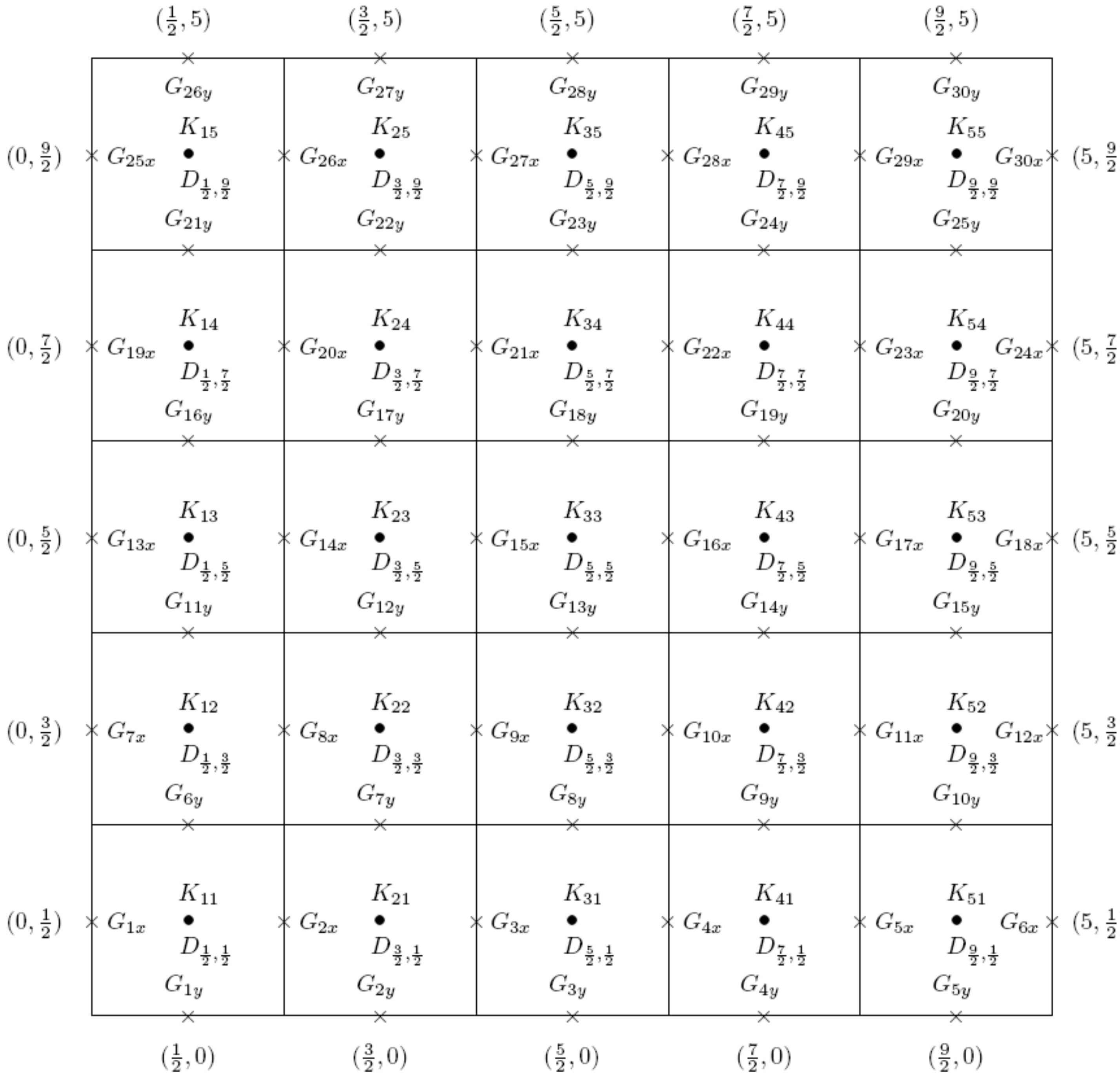
$$B = \begin{bmatrix} -1 & 0 & 0 & 0 & & \cdots & 0 \\ \frac{1}{8} & -\frac{1}{8} & 0 & 0 & & \cdots & 0 \\ -\frac{1}{8} & \frac{1}{8} & 0 & 0 & & \cdots & 0 \\ 0 & 0 & 0 & \ddots & & \cdots & 0 \\ & & & \ddots & 0 & -\frac{1}{8} & \frac{1}{8} \\ & & & & 0 & \frac{1}{8} & -\frac{1}{8} \\ 0 & \cdots & & & 0 & 0 & 1 \end{bmatrix}$$

Second Order Operators: B (Cont'd)

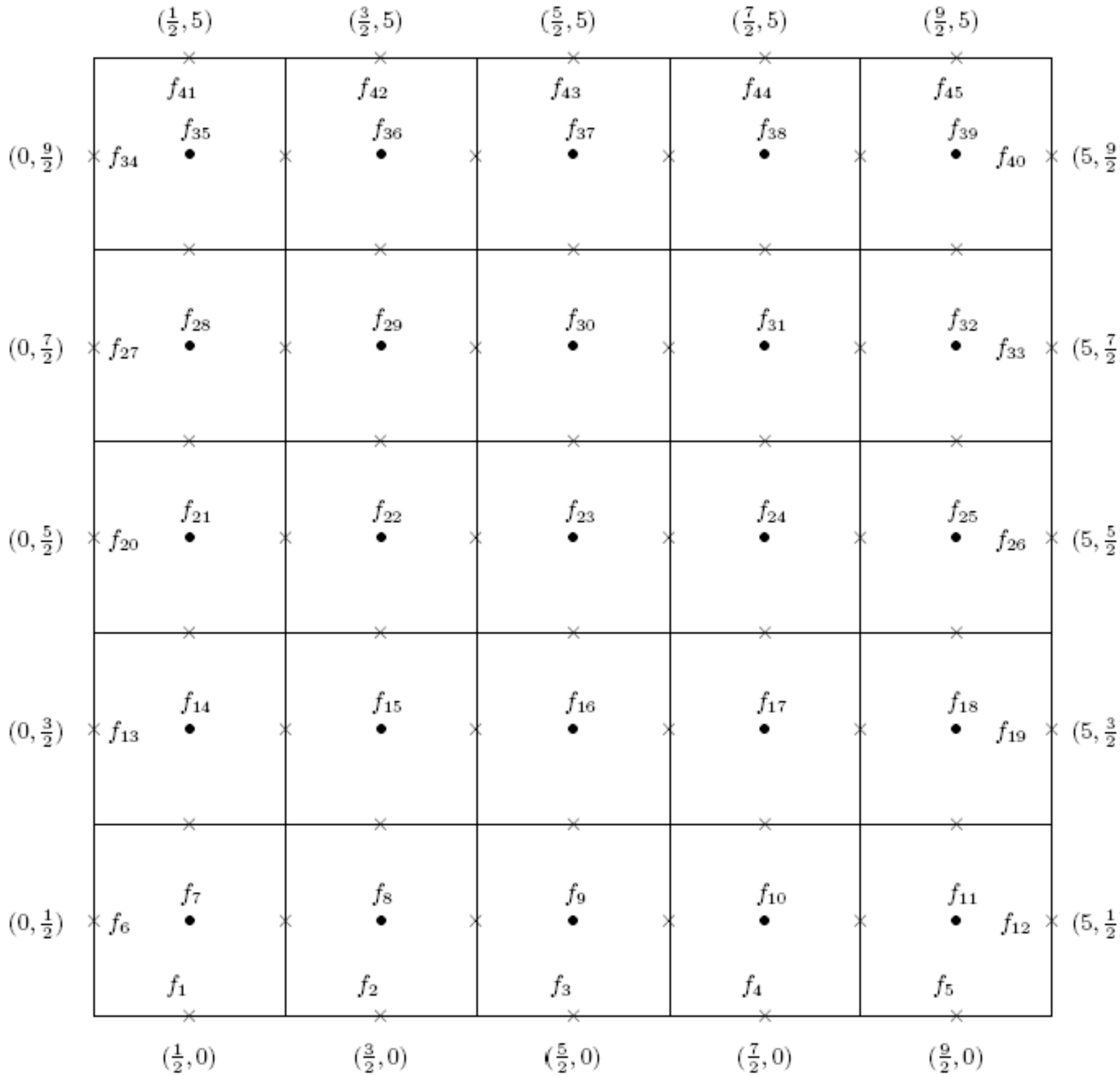
$$B = QD + (PG)^{\mathrm{T}}$$

$$P = \mathrm{diag}\left(\tfrac{3}{8}, \quad \tfrac{9}{8}, \quad 1 \quad \cdots \quad 1, \quad \tfrac{9}{8}, \quad \tfrac{3}{8}\right)$$

$$Q = I$$



$(\frac{1}{2},5)$ $(\frac{3}{2},5)$ $(\frac{5}{2},5)$ $(\frac{7}{2},5)$ $(\frac{9}{2},5)$
G_{26y} G_{27y} G_{28y} G_{29y} G_{30y}
K_{15} K_{25} K_{35} K_{45} K_{55}
$(0,\frac{9}{2})$ G_{25x} G_{26x} G_{27x} G_{28x} G_{29x} G_{30x} $(5,\frac{9}{2})$
$D_{\frac{1}{2},\frac{9}{2}}$ $D_{\frac{3}{2},\frac{9}{2}}$ $D_{\frac{5}{2},\frac{9}{2}}$ $D_{\frac{7}{2},\frac{9}{2}}$ $D_{\frac{9}{2},\frac{9}{2}}$
G_{21y} G_{22y} G_{23y} G_{24y} G_{25y}
K_{14} K_{24} K_{34} K_{44} K_{54}
$(0,\frac{7}{2})$ G_{19x} G_{20x} G_{21x} G_{22x} G_{23x} G_{24x} $(5,\frac{7}{2})$
$D_{\frac{1}{2},\frac{7}{2}}$ $D_{\frac{3}{2},\frac{7}{2}}$ $D_{\frac{5}{2},\frac{7}{2}}$ $D_{\frac{7}{2},\frac{7}{2}}$ $D_{\frac{9}{2},\frac{7}{2}}$
G_{16y} G_{17y} G_{18y} G_{19y} G_{20y}
K_{13} K_{23} K_{33} K_{43} K_{53}
$(0,\frac{5}{2})$ G_{13x} G_{14x} G_{15x} G_{16x} G_{17x} G_{18x} $(5,\frac{5}{2})$
$D_{\frac{1}{2},\frac{5}{2}}$ $D_{\frac{3}{2},\frac{5}{2}}$ $D_{\frac{5}{2},\frac{5}{2}}$ $D_{\frac{7}{2},\frac{5}{2}}$ $D_{\frac{9}{2},\frac{5}{2}}$
G_{11y} G_{12y} G_{13y} G_{14y} G_{15y}
K_{12} K_{22} K_{32} K_{42} K_{52}
$(0,\frac{3}{2})$ G_{7x} G_{8x} G_{9x} G_{10x} G_{11x} G_{12x} $(5,\frac{3}{2})$
$D_{\frac{1}{2},\frac{3}{2}}$ $D_{\frac{3}{2},\frac{3}{2}}$ $D_{\frac{5}{2},\frac{3}{2}}$ $D_{\frac{7}{2},\frac{3}{2}}$ $D_{\frac{9}{2},\frac{3}{2}}$
G_{6y} G_{7y} G_{8y} G_{9y} G_{10y}
K_{11} K_{21} K_{31} K_{41} K_{51}
$(0,\frac{1}{2})$ G_{1x} G_{2x} G_{3x} G_{4x} G_{5x} G_{6x} $(5,\frac{1}{2})$
$D_{\frac{1}{2},\frac{1}{2}}$ $D_{\frac{3}{2},\frac{1}{2}}$ $D_{\frac{5}{2},\frac{1}{2}}$ $D_{\frac{7}{2},\frac{1}{2}}$ $D_{\frac{9}{2},\frac{1}{2}}$
G_{1y} G_{2y} G_{3y} G_{4y} G_{5y}
$(\frac{1}{2},0)$ $(\frac{3}{2},0)$ $(\frac{5}{2},0)$ $(\frac{7}{2},0)$ $(\frac{9}{2},0)$



$(\frac{1}{2},5)$
$(\frac{3}{2},5)$
$(\frac{5}{2},5)$
$(\frac{7}{2},5)$
$(\frac{9}{2},5)$
f_{41} f_{42} f_{43} f_{44} f_{45}
f_{35} f_{36} f_{37} f_{38} f_{39}
$(0,\frac{9}{2})$ f_{34} f_{40} $(5,\frac{9}{2})$
f_{28} f_{29} f_{30} f_{31} f_{32}
$(0,\frac{7}{2})$ f_{27} f_{33} $(5,\frac{7}{2})$
f_{21} f_{22} f_{23} f_{24} f_{25}
$(0,\frac{5}{2})$ f_{20} f_{26} $(5,\frac{5}{2})$
f_{14} f_{15} f_{16} f_{17} f_{18}
$(0,\frac{3}{2})$ f_{13} f_{19} $(5,\frac{3}{2})$
f_7 f_8 f_9 f_{10} f_{11}
$(0,\frac{1}{2})$ f_6 f_{12} $(5,\frac{1}{2})$
f_1 f_2 f_3 f_4 f_5
$(\frac{1}{2},0)$
$(\frac{3}{2},0)$
$(\frac{5}{2},0)$
$(\frac{7}{2},0)$
$(\frac{9}{2},0)$

Example 1: Dirichlet BCs

- BVP:

$$-div\left(K\, grad\, u\right) = F\left(x, y\right) \quad \text{on} \quad \left(0,1\right)\times\left(0,1\right)$$

$$F\left(x,y\right) = -2(1+x^2+xy+y^2)e^{xy}$$

- Tensor:

$$K = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

- Exact Solution:

$$u\left(x,y\right) = e^{xy}$$

Taken from: **"The Numerical Solution of Diffusion Problems in Strongly Heterogeneous Non-isotropic Materials"** by James Hyman, Mikhail Shashkov, and Stanly Steinberg.

Results

	Mimetic Method		Support Operator Method	
n	**Mean Sqr Err**	**Max Err**	**Mean Sqr Err**	**Max Err**
3	1.89E-02	3.57E-02		
4	1.08E-02	2.25E-02		
5	7.00E-03	1.54E-02		
10	1.80E-03	4.40E-03		
17	6.21E-04	1.60E-03	1.06E-03	3.74E-03
20	4.49E-04	1.20E-03		
33	1.66E-04	4.48E-04	2.58E-04	9.66E-04
65	4.29E-05	1.18E-04	6.36E-05	2.45E-04

Example 2: Dirichlet BCs

- BVP:

$$-div\left(K\, grad\, u\right) = F\left(x, y\right) \quad \text{on} \quad \left(0,1\right)\times\left(0,1\right)$$

$$F\left(x,y\right) = -\left[-22\left(y-y^2\right)-26\left(x-x^2\right)+18\left(1-2x\right)\left(1-2y\right)\right]$$

- Tensor:

$$K = \begin{bmatrix} 11 & 9 \\ 9 & 13 \end{bmatrix}$$

- Exact Solution:

$$u\left(x,y\right) = \left(x-x^2\right)\left(y-y^2\right)$$

Taken from: **Mixed Finite Elements for Elliptic Problems with Tensor Coefficients as Cell-Centered Finite Differences.** Todd Arbogast, Mary F. Wheeler, and Ivan Yotov.

Results

	Mimetic Method		Uniform cell-centered Method	
n	Mean Sqr Err	Max Err	Mean Sqr Err	Max Err
3	4.80E-03	7.80E-03	2.60E-01	
4	3.00E-03	5.40E-03	1.46E-01	
5	2.00E-03	3.90E-03	9.36E-02	
10	5.73E-04	1.20E-03	2.34E-02	
17	2.08E-04	4.43E-04	8.09E-03	
20	1.51E-04	3.25E-04	5.85E-03	
33	5.67E-05	1.23E-04	2.15E-03	
65	1.48E-05	3.31E-05	5.54E-04	

2-D Elastic Propagation With a Free Surface BC (Otilio Rojas)

Equation for the displacement vector field **u**(x,t):

$$\rho \frac{\partial^2}{\partial t^2} \boldsymbol{u} = div \cdot (\boldsymbol{\tau} + \boldsymbol{m})$$

Stress-Strain Relations define the symmetric stress tensor $\boldsymbol{\tau}$:

$$\boldsymbol{\tau} = \lambda (div\,\mathbf{u})\mathbf{I} + \mu \left(grad(\mathbf{u}) + grad(\mathbf{u})^{\mathrm{T}} \right)$$

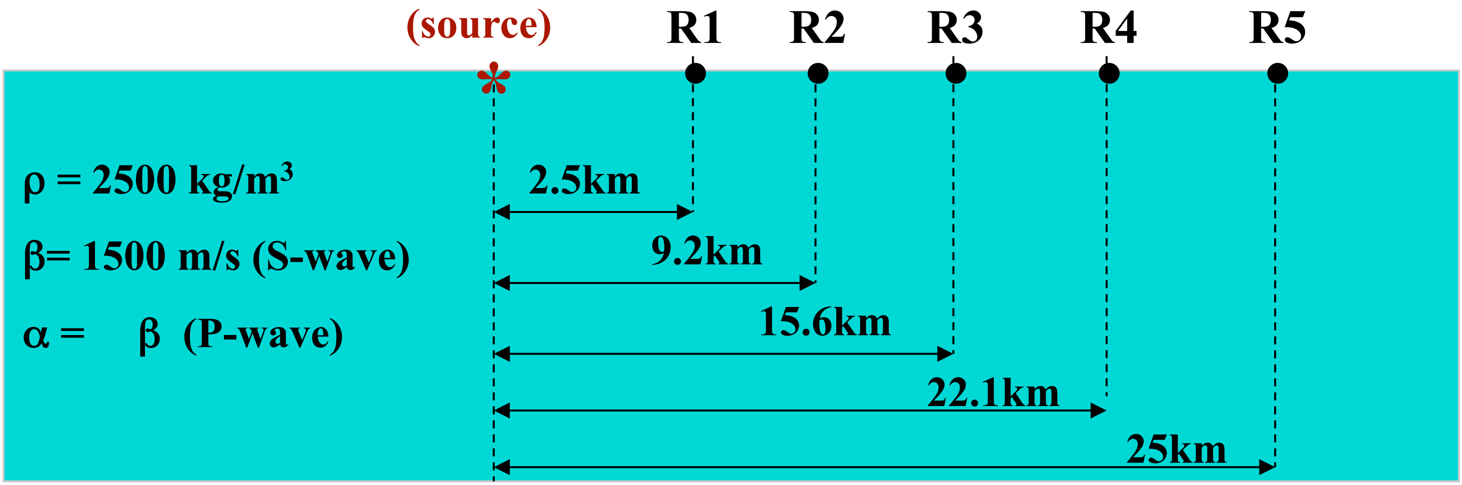
Parameters:

$\lambda(x), \mu(x)$	Lame's constants
$\rho(x)$	Density of the elastic medium
$m(x,t)$	Moment tensor of source distribution

Free-Surface BC:

$$\tau \cdot \vec{n} = 0$$

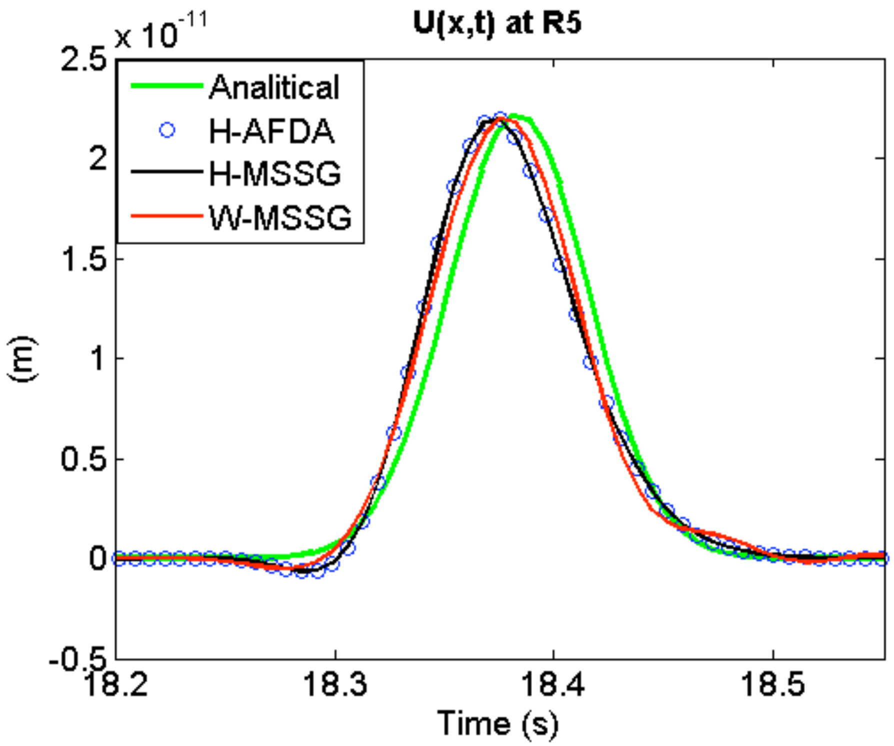
Test Case: Lamb's Problem

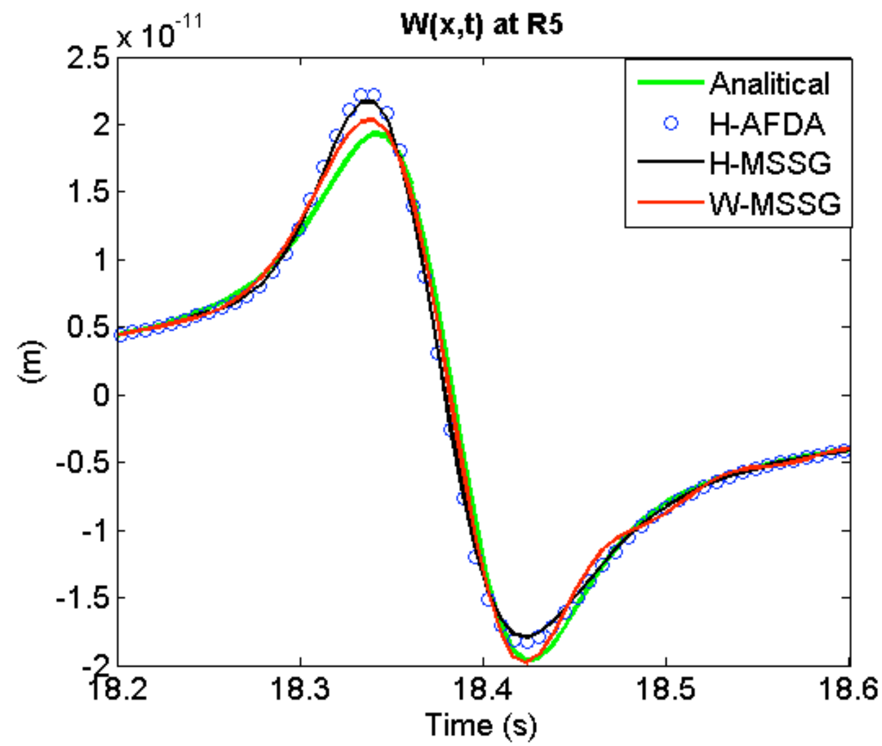


Lamb's Problem → Rayleigh waves are strongly excited

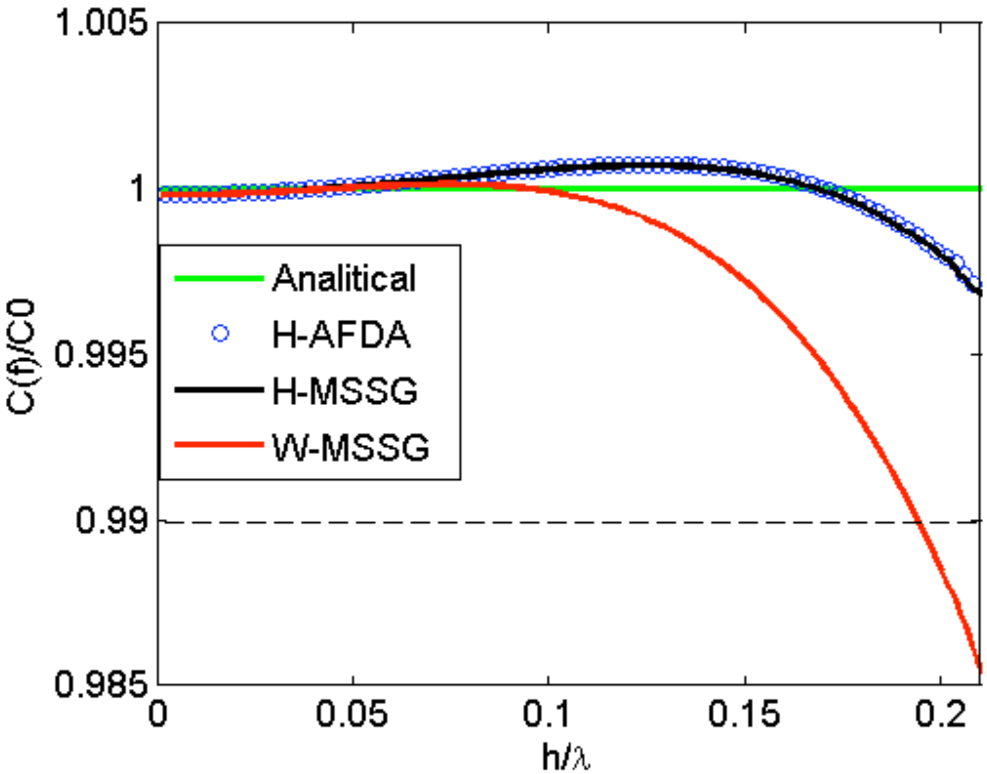
→ Analitical solution in closed form (Cagniard-De Hoop's technique)

Results From Mimetic Schemes: H-MSSG, W-MSSG



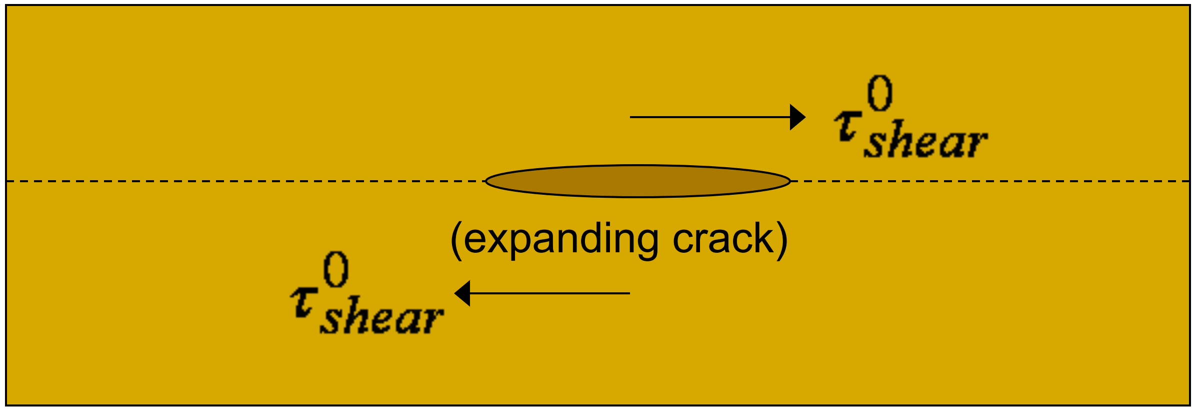


Dispersion Analysis: Rayleigh Wave



Scheme	Points per λ_R			
	σ = 0.20	σ = 0.25	σ = 0.30	σ = 0.35
H-AFDA	5.1	4.9	5.3	5.2
H-MSSG	**4.8**	**4.8**	**4.9**	**4.9**
W-MSSG	5.1	5.2	5.2	5.2

Spontaneous Rupture in Elastic Media



BCs: Spontaneous shear ruptures

$$\begin{cases} \tau - \tau_c \le 0 \\ \vec{\boldsymbol{\tau}}\,\dot{s} - \tau_c\,\dot{\vec{\mathbf{s}}} = 0 \end{cases} \Longrightarrow \begin{cases} \dot{s}\left(\tau - \tau_c\right) = 0 \\ \dot{\vec{\mathbf{s}}} = \dot{s}\,\vec{\boldsymbol{\tau}}/\tau_c \end{cases}$$

(Shear traction doesn't exceed frictional strength)

(Slip rate non-zero if $\tau < \tau c$)

(Colinearity of traction and slip rate)

(Day, 2005: JGR)

(Non-linear mixed BVP with an evolving boundary)

Constitutive Equations (Friction Laws)

$$\tau_c = F(s, \dot{s}, \sigma_n^{eff}, c_e, \lambda_e, T, \Psi)$$

Slip-dependent:

$$F = F(s)$$

Accounts for reduction of τc as slip increases (abrasion of asperities)

Rate- and state-dependent:

$$F = F(\dot{s}, \sigma_n^{eff}, \Psi)$$

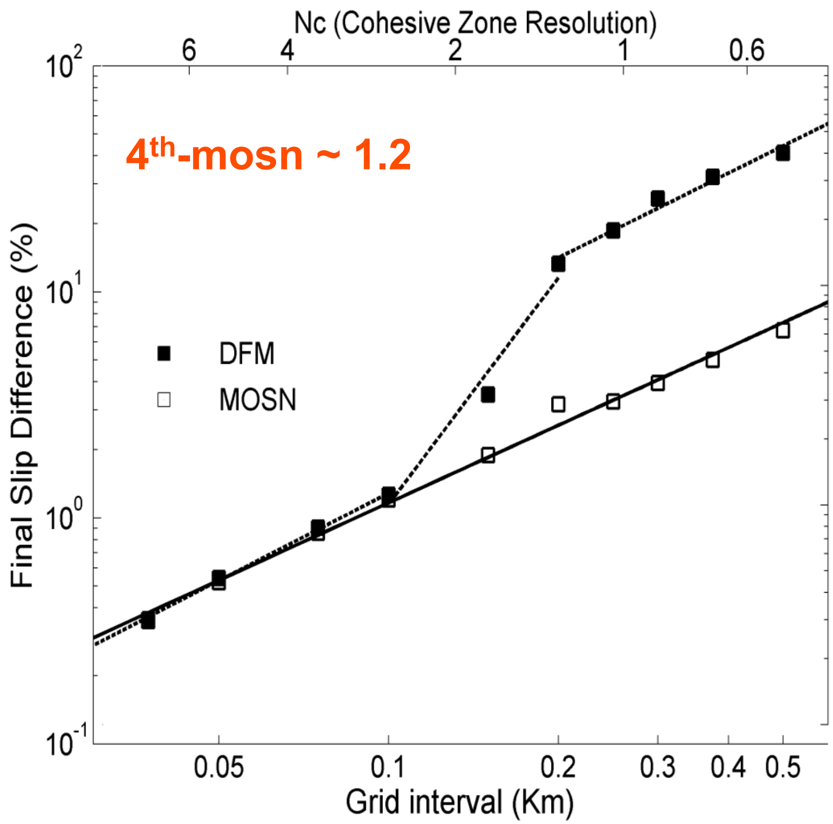
Explain stick-slip behaviors seen in experiments

τ_c	maximum frictional strength on the fault
	effective normal stress – includes effects of pore fluids
C_e	chemical effect of the fluid pressure
λ_c	geometric characteristics of the fault surface (roughness)

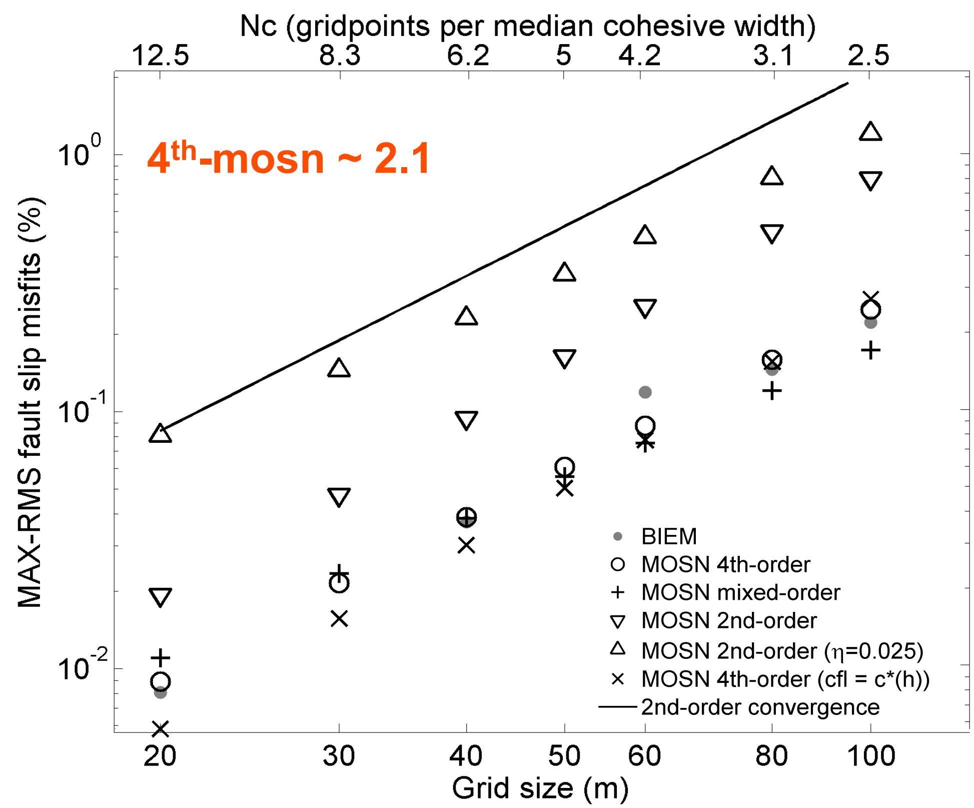
s	slip
	slip rate
T	temperature
Ψ	vector of state variables

Convergence: Fault Slip

Slip-dependent:



Rate-and-state friction:



4th Order MFD Yields
(Rojas et al, 2008-2009)

- Efficient modeling of Rayleigh-Pulse propagation along free surfaces
- Higher accuracy than second order solutions to rupture propagation problems
- Second-order convergence of rupture simulations on rate-and-state interfaces

Non-Uniform Operators G and D

- Operators G and D
  - Develop local transformations of the cells by using a "reference set of cells" in order to obtain local, mimetic operators G and D for non-uniform meshes.
- The Reference Set of Cells: RSC
  - The RSC is a set with two objects: one cell, called CD, for the construction of the divergence operator and two juxtaposed, uniform cells, called CG, for the construction of the gradient operator.

Weight Operators P and Q and Boundary Operator B (David Batista)

- Weight P and Q
  - The weight matrices for the generalized inner products are calculated based on a discrete version of the fundamental theorem of line integrals and a discrete Green identity that calculates the flux across the boundary.
- Boundary operator B
  - From the Stokes' theorem we get an expression for B, which is

$$B = \hat{D}^T Q + P G$$

2^{nd} Order Non-Uniform Operators

$$G = \begin{bmatrix} \frac{1}{J_{x_0}} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{J_{x_1}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{J_{x_2}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{J_{x_3}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{J_{x_4}} \end{bmatrix} \begin{bmatrix} -\frac{8}{3} & 3 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & -3 & \frac{8}{3} \end{bmatrix}$$

2^{nd} Order Non-Uniform Operators (Cont'd)

$$D = \begin{bmatrix} \frac{1}{J_{x_{1/2}}} & 0 & 0 & 0 \\ 0 & \frac{1}{J_{x_{3/2}}} & 0 & 0 \\ 0 & 0 & \frac{1}{J_{x_{5/2}}} & 0 \\ 0 & 0 & 0 & \frac{1}{J_{x_{7/2}}} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

2^{nd} Order Non-Uniform Operators

- Weights for inner products

$$Q_{nu} = \hat{J}_D^{-1} Q_u, \qquad P_{nu} = P_u J_G^{-1}$$

- This technique produces the following result:

$$B_{nu} = \hat{D}_{nu}{}^T Q_{nu} + P_{nu} G_{nu} = B_u$$

Continuum Problem

$$-\nabla^2 f(x) = F(x), \quad x \in [0,1]$$

$$\begin{cases} f(0) - f'(0) = b_1 \\ f(1) + f'(1) = b_2 \end{cases}$$

$$F(x) = \frac{2\times 10^6 x}{\arctan(100)\left(1 + 1\times 10^4 x^2\right)^2}$$

$$b_1 = \frac{100}{\arctan(100)}, \quad b_2 = 1 + \frac{100}{\arctan(100)\left(1+1\times 10^4\right)}$$

Discrete Equations

$$\begin{cases} f(0) - f'(0) = b_1 \\ f(1) + f'(1) = b_2 \end{cases}$$

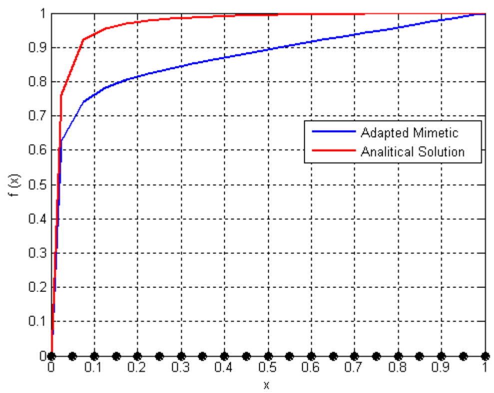
$$-\nabla^2 f(x) = F(x)$$

$$A f + B G f = \begin{pmatrix} b_1 & 0 & \cdots & 0 & b_2 \end{pmatrix}^{\mathrm{T}} = \vec{b}$$

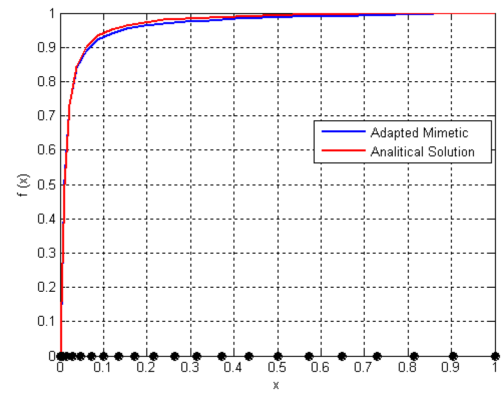
$$\hat{D} G = F$$

$$\left(\alpha A + \beta B G + \hat{D} G \right) f = F + \vec{b}$$

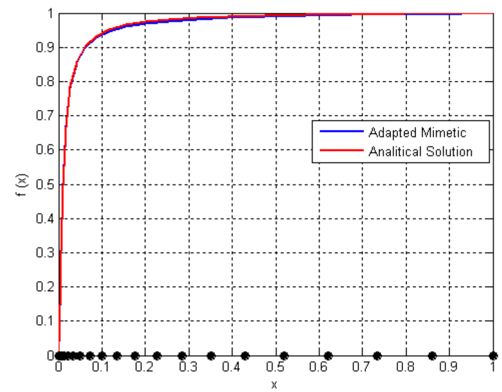
Numerical Results



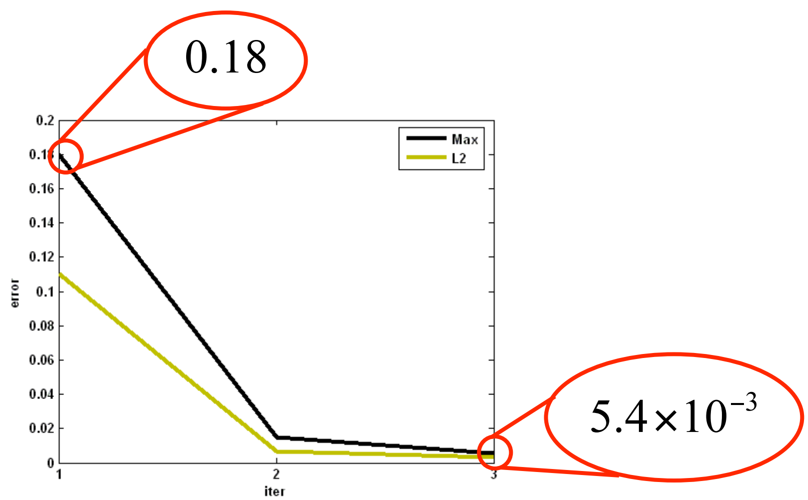
Uniform Mesh



First adaptation



Second adaptation



Mesh size function

$$\frac{1}{\| G\, f \| + a} + b$$

Convergence Analysis

2nd order operators

$Scheme$	$E_h - -\|\cdot\|_\infty$	$E_h - -\|\cdot\|_{L_2}$
$Uniform$	$112.6 \times h^{1.88}$	$129.4 \times h^{2.11}$
$Adapted$	$1.3 \times h^{1.97}$	$1.1 \times h^{2.04}$
$Random$	$19.82 \times h^{1.14}$	$19.0 \times h^{1.32}$

4th order operators

$Scheme$	$E_h - -\|\cdot\|_\infty$	$E_h - -\|\cdot\|_{L_2}$
$Uniform$	$406438 \times h^{3.71}$	$665340 \times h^{3.99}$
$Adapted$	$119.4 \times h^{4.11}$	$44.3 \times h^{4.17}$
$Random$	$152.5 \times h^{1.52}$	$108.5 \times h^{1.52}$

Convergence Analysis (Cont'd)

$$E = c\,h^{p} + O\left(h^{p+1}\right)$$

$$\left\| \hat{f} - f \right\|_{\infty} = \max\left\{ \left| \hat{f}_{i+.5} - f_{i+.5} \right|, \quad i = 1, \ldots, n \right\}$$

$$\left\| \hat{f} - f \right\|_{L_2} = \sqrt{\sum_{i=1}^{n-1} \left(\hat{f}_{i+.5} - f_{i+.5} \right)^{2} \left(J_D^{-1} \right)_{i,i}}$$

2-D Boundary Layer Problem

$$-\nabla^2 f(x,y) = F(x,y) \qquad (x,y) \in [0,1] \times [0,1]$$

$$F(x,y) = \frac{2\times10^6}{\arctan(100)}\left[\frac{x}{(1+1\times10^4 x^2)^2} + \frac{y}{(1+1\times10^4 y^2)^2}\right]$$

$$\alpha\, f(0,y) + \beta\, f'(0,y) = \frac{\arctan(100y)}{\arctan(100)}$$

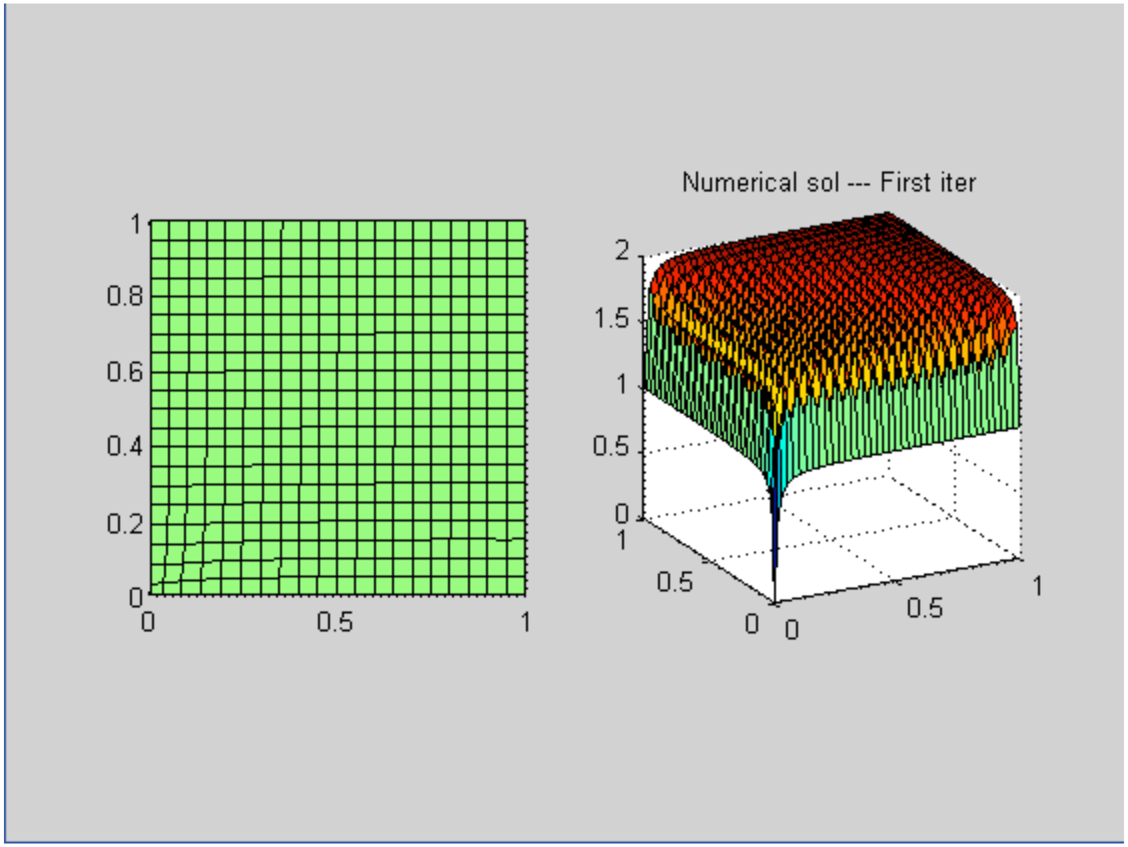
$$\alpha\, f(1,y) + \beta\, f'(1,y) = 1 + \frac{\arctan(100y)}{\arctan(100)}$$

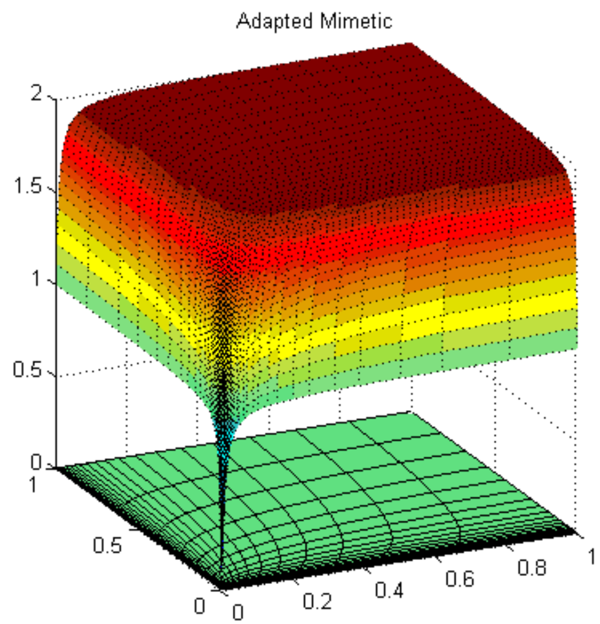
$$\alpha\, f(x,0) + \beta\, f'(x,0) = \frac{\arctan(100x)}{\arctan(100)}$$

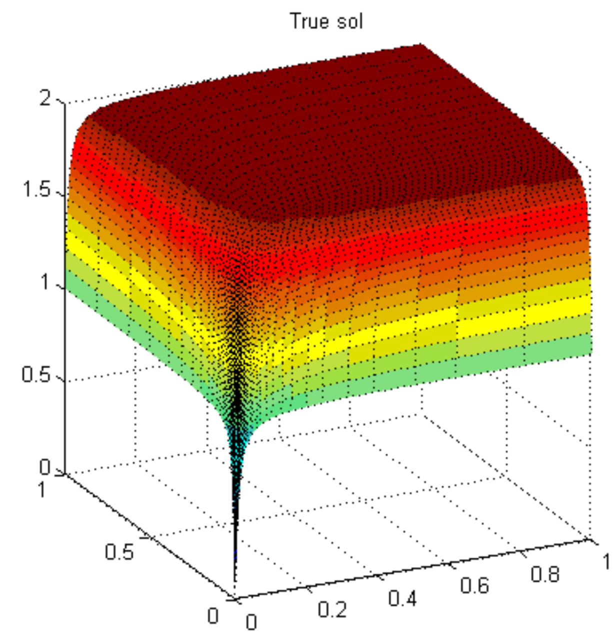
$$\alpha\, f(x,1) + \beta\, f'(x,1) = 1 + \frac{\arctan(100x)}{\arctan(100)}$$

$$\alpha = 1 \qquad \beta = 0$$

2-D Boundary Layer Problem (Cont'd)




Adapted Mimetic
2
1.5
1
0.5
0
1
0.5
0
0
0.2
0.4
0.6
0.8
1


True sol
2
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1
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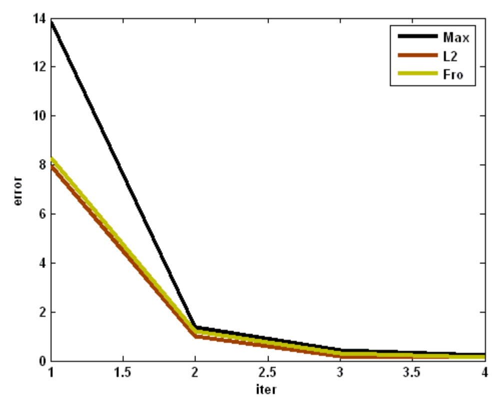
0
0
0.2
0.4
0.6
0.8
1


Max
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8
6
4
2
0
1
1.5
2
2.5
3
3.5
4

Current and Future Work

- 3-D/General grids
- Curl operator

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