# *Wave Breaking Dissipation*

## *and Wave/Current Interactions*

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# Uncertainty Quantification Group

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  - Shankar Venkataramani (Math)
  - Kevin Lin (Math)
  - Rabi Bhattacharya (Math, Statistics)
  - Kobus Barnard (Computer Science)
  - Hermann Flaschka (Math)
- Post-Docs
  - P. Krause
  - J. Ramírez
  - P. Dostert
  - T.-T. Shieh
- Graduate Students: 15, Undergraduate Students: 2.

# Focus Problems

Weather Hydrogeology Vision/Structure

Estimation, Sampling, Dynamical Systems, Machine-Learning, Computer Grammars, Large-scale Computing, Statistical Analysis, Data Management.

UQG

# Wave Breaking Dissipation Team

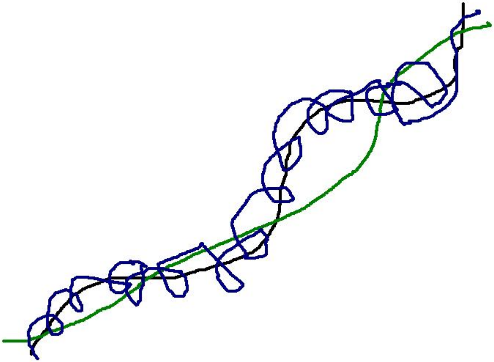
## COLLABORATORS

| | |
|---|---|
| Jorge Ramírez & Brad Weir | University of Arizona |
| Emily Lane | NIWA, New Zealand |
| Jim McWilliams | UCLA |
| Michael Banner | UNSW, Australia |

# Lagrangian/Eulerian Projections in Multiscale Setting

## THEME

Multiscale projection of systems of (mostly) hyperbolic equations between the Eulerian and Lagrangian frames.



## Outcomes Relevant to Ocean Dynamics:

- Wave/current interaction theory (Vortex Force)
- Stress and pressure-gradient adjustments due to the radiation stress
- Interactions of waves and currents in littoral current instabilities
- Interaction of waves and currents in rip current instabilities
- The rip surge, a different type of rip current
- Origin and maintenance of shore-face connected sand ridges
- Oceanic Lagrangian data assimilation
- *Inclusion of Dissipative Mechanisms via stochastic parametrization (wave breaking, bottom drag)*

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# Transport Due to Oscillatory Flow

$$\frac{d\mathbf{X}}{dt} = \mathbf{V}$$
$$\mathbf{X}(0) = \mathbf{X}_0$$

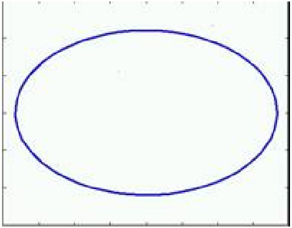


Figure: Particle Motion Under Linear Waves

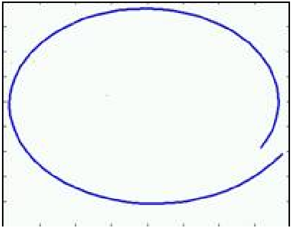
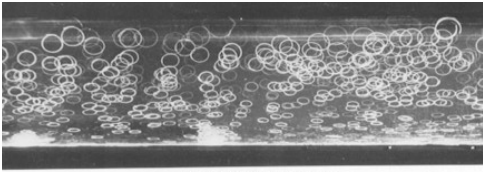
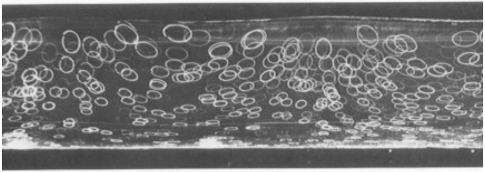


Figure: Particle Motion Under Nonlinear Waves

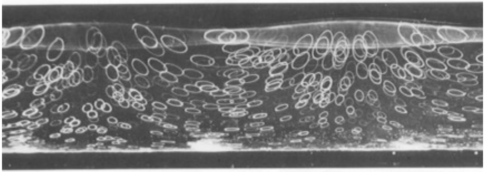
## The Basic Research Problems



No reflection: pure progressive waves



24% reflection



38% reflection

# Lagrangian Motion Under White Capping

$$d\mathbf{X} = \mathbf{V}dt$$

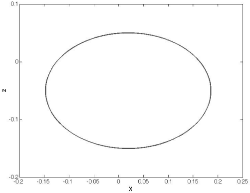



Figure: Deterministic

# Lagrangian Motion Under White Capping

$$d\mathbf{X} = \mathbf{V}dt$$

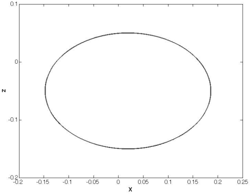



Figure: Deterministic

# Lagrangian Motion Under White Capping

$$d\mathbf{X}_t = \mathbf{V}(\mathbf{X},t)dt + d\mathbf{W}_t(\mathbf{X},t)$$

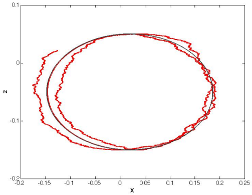
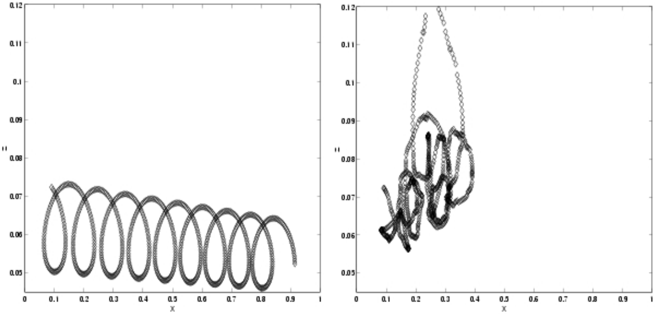



Figure: Stochastic

K. M. Janson & G. D. Lythe, PRL **81**,1998




J. Restrepo and G. Leaf *Wave-generated Transport Induced by Ideal Waves* J. Phys. Oceanogr. 2002

# What Stochastic Model?

$$d\mathbf{X}_t = \mathbf{V}(\mathbf{X},t)dt + d\mathbf{W}_t(\mathbf{X},t)$$

Experiments are needed to determine the right model

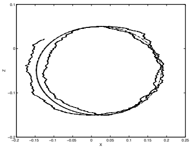


Figure: $dW$ standard white noise

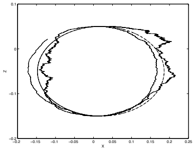


Figure: $d\mathbf{W}$ with added mean-reverting process

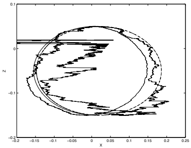


Figure: $d\mathbf{W}$ with added jump process

# Why a Stochastic Model?

- Ideal way to parameterize some aspects of small scale motions
- Parameterization permits stronger connection to field data

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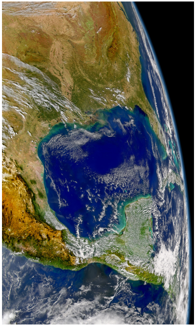
## Main Goals

- How does dissipation at wave scales manifest itself at longer time scales?
- Can we find the right stochastic parameterization?
- Can we parameterize dissipation in such a way that we use global information more efficiently? (Lagrangian/Eulerian)

# Scale Range of the Model

- 10 secs-months
- 100m-basin scale
- Speed: waves $>$ currents
- $kH \sim 1$
- Applications:
  - climate dynamics (transport)
  - erodible bed dynamics
  - river plume evolution
  - algal/plankton blooms
  - pollution



J. McWilliams and J. M. Restrepo *The Wave-driven OceanCirculation* J. Phys. Oceanogr. (1999)
J. Restrepo *Wave-Current Interactions and Shore-connected Bars* J. Estuarine Sci. (2001)
J. McWilliams J. M. Restrepo, E. Lane *An asymptotic Theory for the Interaction of Waves and Currents in Coastal Waters* J. Fluid Mechanics (2004)
E. Lane, J. M. Restrepo, J. McWilliams *Wave-Current Interaction: A comparison of radiation-stress and vortex-force representations* J. Phys Oceanogr (2007)

# Conservative Wave/Current Model

GOALS:

- Formulate a model on the larger spatio-temporal scales that accurately captures the dynamics of the small scales *without* requiring solving for these on the smaller scales.
- Describe *how* waves affect currents and vice versa.
- The model should be extensible, to eventually include:
  - tracer dynamics.
  - wave generation.
  - dissipation in the bulk, due to breaking events, due to bottom drag.
  - wind forcing.

# Conservative Wave/Current Model

STRATEGY

- Use a quasi geostrophic flow model or a shallow water wave model
- Introduce three scales: waves $\mathbf{u}^w$, long-waves $\mathbf{u}^{lw}$, currents $\mathbf{v}$.
- Use asymptotics balances with waves of comparable or larger amplitude than currents
- Use time-averaging and projections to derive filtered equations

# Multi-scale Methodology

Momentum, Continuity, and Tracer Equations:

$$\begin{aligned}\mathbf{q}_t+(\mathbf{q}\cdot\nabla)\mathbf{q} &+ 2\Omega\times\mathbf{q}-\tilde{b}\hat{\mathbf{z}}+\frac{1}{\rho_0}\nabla\tilde{p} = \nu\nabla^2\mathbf{q},\\ \nabla\cdot\mathbf{q} &= 0,\end{aligned}$$

$$\frac{D\tilde{\theta}}{Dt} = \kappa\nabla^2\tilde{\theta},$$

The surface boundary conditions at $z=\eta(\mathbf{x}_h,t)$ are the following:

$$\begin{aligned} w &= \frac{D\eta}{Dt}, \quad \tilde{p} = g\rho_0\eta+\tilde{p}_a,\\ \nu\frac{\partial\mathbf{q}}{\partial z} &= \frac{1}{\rho_0}\tau, \quad \kappa\frac{\partial\tilde{\theta}}{\partial z} = \mathscr{T}.\end{aligned}$$

Linear Gravity Waves:

$$
\begin{aligned}
\mathbf{u}_k^w &= -\nabla \varphi_k^w, \\
\varphi_k^w &= -\frac{e^{kz}}{k} \frac{\partial \eta_k^w(\mathbf{x}_h, t)}{\partial t}, \\
\eta_k^w &= a_k \cos[\mathbf{k}_h \cdot \mathbf{x}_h - \sigma_k t],
\end{aligned}
$$

$$\mathbf{q} = \varepsilon[\mathbf{u}^w(\mathbf{x},t) + \varepsilon\mathbf{v}(\mathbf{x},t_s,t)],$$

where $\omega = \nabla \times \mathbf{q} = \nabla \times \mathbf{v}$

$$\frac{\partial \omega}{\partial t} + \varepsilon^2 \frac{\partial \omega}{\partial t_s} = \nabla \times [\varepsilon(\mathbf{u}^w + \varepsilon\mathbf{v}) \times 2\Omega] + \nabla \times [\varepsilon(\mathbf{u}^w + \varepsilon\mathbf{v}) \times \omega] \\ \varepsilon^2 \nu \nabla^2 \omega + \varepsilon^2 \nabla \times \hat{\mathbf{z}}\, b,$$

Lowest orders in vorticity:

$$\omega_0 = \omega_0(\mathbf{x},t_s).$$
$$\omega_1 = \nabla \times (\mathbf{U} \times \mathbf{Z}),$$

where $\mathbf{U} \equiv \int^t \mathbf{u}^w(\cdot,s)\,ds$ and $\mathbf{Z} = \omega + 2\Omega.$

The next order up yields the wave/current equation, after time averaging over waves:

$$\frac{\partial \mathbf{v}_0}{\partial t_s} - \mathbf{V} \times \mathbf{Z} + \nabla \Phi - \ b_0 \hat{\mathbf{z}} \ = \ \nu \nabla^2 \mathbf{v}_0,$$

where we define

$$\Phi \ = \ p_0 + \frac{1}{2} \mathbf{V}^2.$$

$$\mathbf{V} = \mathbf{v} + \mathbf{u}^{St}, \qquad \mathbf{Z} = \omega + 2\Omega$$

where $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}'$, where $< \mathbf{v} >= \mathbf{v}_0$. The Stokes Drift Velocity is

$$\mathbf{u}^{St} = \left\langle \int^t \mathbf{u}^w(\mathbf{x}, s)\, ds \cdot \nabla \mathbf{u}^w \right\rangle.$$

The tracer equation is

$$\frac{\partial C_0}{\partial t_s} + \mathbf{V} \cdot \nabla C_0 \ = \ \kappa \nabla^2 C_0.$$

# Boundary Conditions

The surface boundary conditions at $z = \eta(\mathbf{x}_h, t)$ are the following:

$$w = \frac{D\eta}{Dt}, \quad \tilde{p} = g\rho_0\eta + \tilde{p}_a, \quad \nu\frac{\partial \mathbf{q}}{\partial z} = \frac{1}{\rho_0}\tau, \quad \kappa\frac{\partial \tilde{C}}{\partial z} = \mathscr{T}.$$

Lead to (at $z = 0$)

$$w_0 = \nabla \cdot \mathbf{M}, \qquad p_0 = \eta_0 + p_{a0} - P$$
$$\nu\left(\frac{\partial \mathbf{v}_0}{\partial z} + \mathbf{S}\right) = \tau, \qquad \kappa\frac{\partial C_0}{\partial z} = \mathscr{T}.$$

where the *wave-induced adjustments* (at $z = 0$) are

$$\mathbf{M} \equiv \left\langle \mathbf{u}^w \eta^w \right\rangle, \quad P \equiv \left\langle p_z^w \eta^w \right\rangle, \quad \mathbf{S} \equiv \left\langle \frac{\partial^2 \mathbf{u}^w}{\partial z^2} \eta^w \right\rangle.$$

# Currents, Shallow Water Wave Case

Momentum

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} + f \hat{\mathbf{z}} \times \mathbf{v} + \nabla p = -\nabla K + \hat{\mathbf{z}} \times \mathbf{J}$$

Continuity

$$\eta_t + \nabla \cdot [H \mathbf{v}] = \nabla \cdot \mathbf{M}$$

Tracer Equation

$$C_t + \mathbf{v} \cdot \nabla C = \kappa \Delta C - \mathbf{u}^{St} \cdot \nabla C.$$

Boundary conditions are modified by the presence of waves and currents

# Waves, Shallow Water Wave Case

The sea elevation is represented by

$$\eta = \sum_j A_j(\mathbf{X},\tau) e^{iS_j(\mathbf{X},\tau)/\varepsilon^2} + c.c$$

where $\mathbf{k}_j = -\nabla_{\mathbf{X}} S_j$, $\omega_j = \partial_\tau S_j$.

$$\begin{aligned} \mathbf{k}_t + \mathbf{C}_g \nabla_{\mathbf{X}} \mathbf{k} &= -\frac{k\sigma}{\sinh[2kH]} \nabla_{\mathbf{X}} H \\ \sigma_t + \mathbf{C}_g \cdot \nabla_{\mathbf{X}} \sigma &= 0 \end{aligned}$$

$\sigma^2 = k \tanh[kH]$ and the group velocity $\mathbf{C}_g = \frac{\sigma}{2k^2}(1 + \frac{2kH}{\sinh[2kH]})\mathbf{k}$.
The amplitude of the waves obey

$$A_t + (\mathbf{C}_g + \mathbf{V}) \cdot \nabla_{\mathbf{X}} A + \frac{1}{2} A \nabla \cdot \mathbf{C}_g + \frac{i}{2} M |A|^2 A + iNA = 0$$

# Currents/Waves and Dissipation due to White Capping

$$dX_t = V(X,t)dt + dW_t(X,t)$$

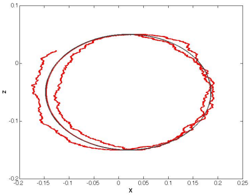



Figure: Stochastic

# Currents/Waves and Dissipation due to White Capping

$$d\mathbf{X}_t = \mathbf{V}(\mathbf{X},t)dt + d\mathbf{W}_t(\mathbf{X},t)$$

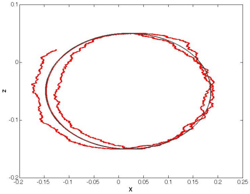



Figure: Stochastic

# Dissipative/Diffusion Current Model

The momentum with dissipation and diffusion:

$$\frac{\partial \mathbf{v}}{\partial T} = \mathbf{V} \times \mathbf{Z} - \nabla\Phi + \langle(\mathbf{B} \times \mathbf{Z}) + \mathbf{b} \times (\nabla \times \mathbf{b}) + [\mathbf{V} \times \nabla \times \mathbf{b}] - \frac{1}{2}\nabla|\mathbf{b}|^2\rangle + \nabla \cdot \mathbf{R}$$

where $\mathbf{V} = \mathbf{v} + \mathbf{u}^{stokes}$, $\mathbf{Z} = \nabla \times \mathbf{v} + 2\Omega$ and $\Phi = p_0 + \frac{1}{2}|\mathbf{V}|^2$.

The dissipation contributes to the vortex force and the Bernoulli head.

The diffusion appears as a viscous term and accounts for boundary layer effects

The tracers obey

$$\frac{\partial C}{\partial T} + \mathbf{V} \cdot \nabla C = -\mathbf{B} \cdot \nabla C + \nabla \cdot \mathbf{Q}.$$

J. M. Restrepo *Wave Breaking Dissipation in the Wave-Driven Ocean Circulation* J. Phys. Oceanogr. (2007)
J.M. Restrepo, J. M. Ramírez, J.C. McWilliams & M. Banner *Wave Breaking Dissipation and Diffusion in Waves and Currents* , in preparation (2008)

Model $\mathbf{b} := \mathbf{B} + \mathbf{b}'$ as the random sum

$$\mathbf{b}(\mathbf{x}_h, z, T) = \sum_{(\mathbf{X}_h,\tau)\in\Phi} \mathbf{b}_{E(\mathbf{X}_h,\tau)}(\mathbf{x}_h - \mathbf{X}_h, z)\,\delta(\tau - T)$$

where

$$\mathbf{b}_E(\mathbf{x}_h - \mathbf{X}_h, z) := \frac{1}{\tau_E}\int_0^{\tau_E} \tilde{\mathbf{b}}_E(\mathbf{x}_h - \mathbf{X}_h, z, t)dt$$

The ensemble average at $(\mathbf{x}_h, z, T)$ of some functional $\mathscr{F}$ of the field $\mathbf{b}$ is:

$$\begin{aligned}
\langle \mathscr{F}(\mathbf{b})\rangle(\mathbf{x}_h, z, T)dT &:= \left\langle \sum_{(\mathbf{X}_h,\tau)\in\Phi} \mathscr{F}(\mathbf{b}_{E(\mathbf{X}_h,\tau)}(\mathbf{x}_h - \mathbf{X}_h, z))\delta(T-\tau) \right\rangle \\
&= \int_{\mathbb{R}}\int_{\mathbf{x}_h-\Omega_E} \mathscr{F}(\mathbf{b}_E(\mathbf{x}_h - \mathbf{X}_h, z))\Lambda(d\mathbf{X}_h, dT)p(E)dE.
\end{aligned}$$

$$\mathbf{B}(z) = \int_{\mathbb{R}}\left[\int_{\tilde{\Omega}_E} \mathbf{b}_E(\mathbf{X}_h, z, T)d\mathbf{X}_h dT\right]\lambda\, p(E)dE$$

# Diffusivity

The observation is that wave breaking increases the size of the mixing layer and this layer will then create a great deal of dissipation.

$$\begin{aligned} \mathbf{R}_v &\approx \nu \frac{\partial \mathbf{v}_h}{\partial z}, \quad \mathbf{R}_h \approx \nu \nabla \mathbf{v}_h \\ Q_v &\approx \kappa \frac{\partial C}{\partial z}, \quad \mathbf{Q}_h \approx \kappa \nabla C. \end{aligned}$$

We assume that

$$\nu \sim \langle \ell_b \left| w^b \right| \rangle, \qquad \kappa \sim \langle \ell_\theta \left| w^b \right| \rangle.$$

$w^b$ is the vertical component of the velocity associated with breaking, and the mixing length is

$$\ell_b = \gamma \eta(\mathbf{x}, t), \qquad \ell_\theta = \alpha \eta(\mathbf{x}, t).$$

# Current Effects on Waves

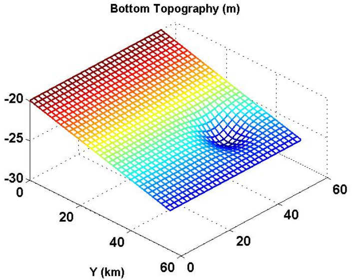



Figure: Bottom Topography

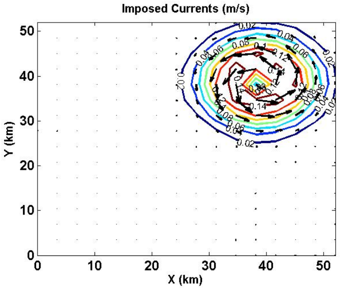



Figure: Current Forcing

# Current Effects on Waves

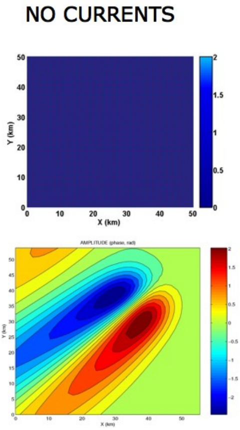



Figure: Waves, No Currents

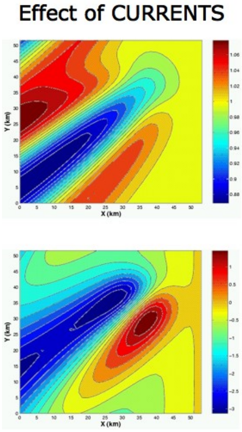



Figure: Waves, With Currents

# Wave Effects on Currents

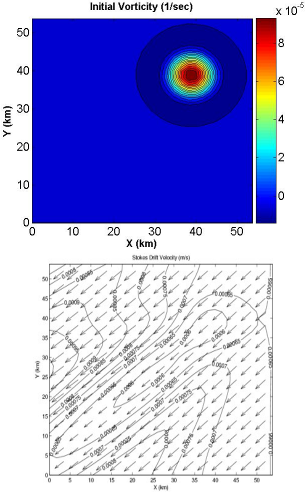



Figure: Initial Current and Waves Forcing

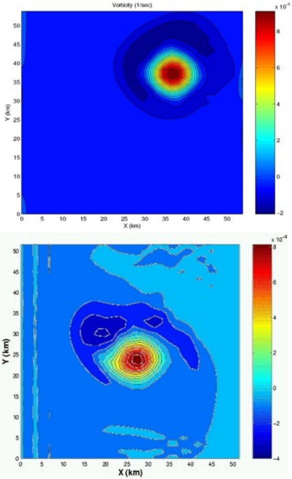



Figure: No Waves, With Waves

# Diffusion and Wave Effects on Currents

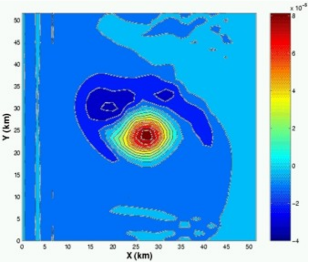



Figure: Without Diffusion

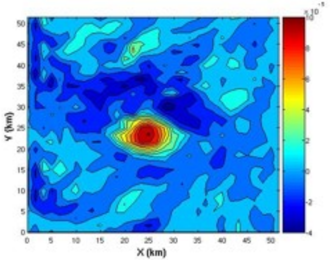



Figure: With Diffusion

# Wave Breaking Dissipation Model

How do you Determine Wave Dissipation (*i.e.* **b**)?

- Determine the actual velocity **b** from field data.
- Use a model.

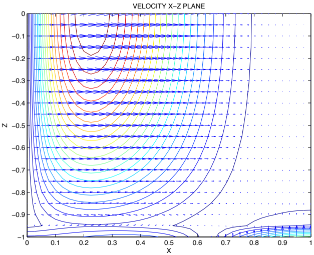
# How do you *Model* Wave Dissipation (*i.e.* **b**)?

Solve

$$\partial_\alpha \mathbf{b} = \frac{1}{Re}\Delta_{\beta,\delta,\gamma}\mathbf{b} + \mathbf{a}\mathscr{X}\mathscr{Y}\mathscr{Z}\mathscr{T}$$

$$\nabla_{\beta,\delta,\gamma}\cdot\mathbf{b} = 0.$$

$$\begin{aligned}
\mathscr{X} &= \beta^3(1-\beta^2), \qquad 0 \le \beta \le 1,\\
\mathscr{Y} &= 1-(\delta-1)^4, \qquad 0 \le \delta \le 2,\\
\mathscr{Z} &= (1-(\gamma-1)^4)^2, \qquad 0 \le \gamma \le\\
\mathscr{T} &= -(1-\alpha^2)(1-e^{5\alpha}), \qquad 0 \le
\end{aligned}$$




$(\alpha,\beta,\delta,\gamma)$ are scaled $(t, x-ct, y, z-\chi ct)$, where $c$ is the wave speed, $\chi c$ is the wave speed in the z direction.

P. Sullivan, J. C. McWilliams, K. Melville, JFM (2004).

# Coupling Wave Groups and Breaking Producing Dissipation

Waves expressed in terms of a carrier and an envelope:

$$\eta(\mathbf{X}_h, t) = \mathrm{Re}\left\{ e^{i(\bar{\mathbf{k}}_h \cdot \mathbf{x}_h + \bar{\sigma} t)} \rho(\mathbf{x}_h, t) e^{i\theta(\mathbf{x}_h, t)} \right\}.$$

where

$$\mathbb{P}(\rho(\mathbf{x}_h, t) \in d\rho) = \frac{\rho}{2\pi\langle\eta^2\rangle} \exp\left\{ -\frac{\rho^2}{2\langle\eta^2\rangle} \right\}$$

a Rayleigh distribution.
The maximum wave group energy is defined as

$$\mu(t) = \max_{x \in \Omega(t)} \rho^2(x, t) k^2$$

The mean growth rate of wave group energy is

$$\delta(t) = \frac{1}{\bar{\sigma}} \frac{\mathrm{D}\mu}{\mathrm{D}t}$$

# 2D Example

(Loading breakmovie)

Figure: Dots indicate breaking events

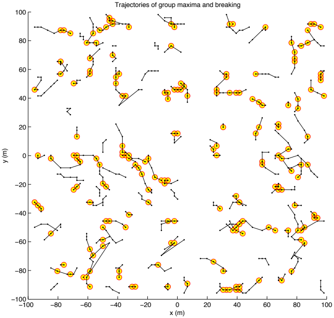



Figure: Breaking Events Corresponding to the Movie.

# 2D Example

(Loading breakmovie)

Figure: Dots indicate breaking events

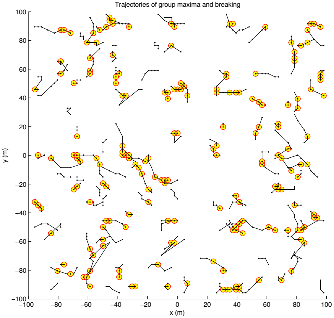



Figure: Breaking Events Corresponding to the Movie.

# Constraining the Breaking Velocity Magnitude and Direction

$$
\begin{aligned}
\frac{\partial \mathbf{v}}{\partial T} &= \mathbf{V}\times\mathbf{Z} - \nabla\Phi \\
&+ \langle(\mathbf{B}\times\mathbf{Z}) + \mathbf{b}\times(\nabla\times\mathbf{b}) + [\mathbf{V}\times\nabla\times\mathbf{b}] - \frac{1}{2}\nabla|\mathbf{b}|^2\rangle \\
&+ \nabla\cdot\mathbf{R}
\end{aligned}
$$

The boundary conditions at $z = 0$ are

$$
\begin{aligned}
w^c(\mathbf{x}_h, 0, T) &= \nabla\cdot\mathbf{M} - w^b(\mathbf{x}_h, 0, T), \\
\mathbf{M} &= \overline{\mathbf{u}^w \eta^w}, \\
\nu\left(\frac{\partial \mathbf{v}}{\partial z} + \mathbf{S}\right\rvert_{z=0} &= \tau - \nu\,\frac{\partial \mathbf{b}}{\partial z}\bigg\rvert_{z=0}, \\
\mathbf{S}(\mathbf{x}_h, t) &= \overline{\frac{\partial^2 \mathbf{u}^w(\mathbf{x}_h, 0, t)}{\partial z^2}\,\eta^w(\mathbf{x}_h, t)}
\end{aligned}
$$

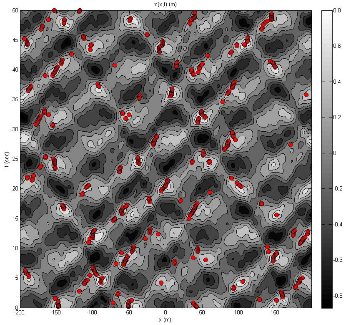



Figure: Breaking events exceeding threshold $\delta$

# Progress Report

- We determined how wave breaking enters wave/current dynamics
  - Wave breaking dissipates waves.
  - Wave breaking affects currents via dissipation and diffusion.
  - Wave breaking affects the effect of residual flow due to waves.
- Proposed how to model breaking event.
  - Wave groups generate probability distribution of breaking events.
  - Can constrain breaking events via wind stress field.
  - Can make a more direct connection to field data (than conventional turbulence ideas).
- Could approach be extended to the problem of wind/wave interactions?

Further information:
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