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### Hopping Behavior and Effects of Noise in Cellular Pattern-Forming Systems

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Abstract: We study the effects of multiplicative noise on a spatio-temporal pattern forming nonlinear Partial Differential Equation (PDE) model for premixed flame instability, known as the Kuramoto-Sivashinsky equation, in a circular domain. Modifications of a previously developed numerical integration scheme allow for longer time integration in the presence of noise. In order to gain additional insight, we focus on a region of parameter space where hopping patterns of the deterministic system arise as well as the region of parameter space where the transition between a single ring to multiple rings of cells appears. We discuss the numerical challenges in the integration of the Kuramoto-Sivashinsky equation in polar coordinates with the addition of the noise term. We also study the effects of additive and multiplicative noise on the normal forms or amplitude equations that describe the dynamics of hopping patterns. Finally we show some results of the implementation of the numerical scheme to solve both the PDE and the normal form equations and discuss preliminary findings.

**Keywords:** pattern formation, noise, normal forms , nonlinear dynamical systems, numerical analysis

#### 1 Introduction

Patterns are found everywhere in nature and can occur in chemical, physical and biological systems. Pattern formation is a phenomenon that has been studied for decades and its analysis very often consists of finding a differential equation that models the space and time evolution of the pattern and coherent structures that lead to their formation. In the particular case of cellular patterns, the coherent structures are driven by symmetry breaking bifurcations, which play a main role in determining the family of structures that can be observed. Moreover, the presence of different types of noise in physical systems has been shown to affect the symmetry breaking bifurcation process. The purpose of our research is the study of different mechanisms that lead to various types of cellular flame patterns. In this work we explore the effects of multiplicative noise in a pattern forming system with two dimensional orthogonal symmetry, O(2)-symmetry, specifically we focus our attention on the effects on hopping patterns which are modulated rotating waves that emerge via a co-dimension three steadystate bifurcation.

#### 2 The Mathematical Model

As mentioned in the introduction, the study of pattern formation often starts with a PDE or ODE that serves as model for the system of study. In our case this model is the Kuramoto-Sivashinsky equation (KS) which is a PDE that describes the spatiotemporal evolution of a flame front of premixed flames [7]. For our current research, we use the following form of the KS equation:

$$\frac{\partial u}{\partial t} = \eta_1 u - (1 + \nabla^2)^2 u - \eta_2 (\nabla u)^2 - \eta_3 u^3 + \xi_1(x, t)u + \xi_2(x, t),$$

where u = u(x, t) represents the perturbation of the planar front (typically assumed to be a flame front) in the direction of propagation,  $\eta_1$  measures the strength of the perturbation force,  $\eta_2$  is a parameter associated with growth in the direction normal to the domain, the term  $\eta_3$  is added to temper the bifurcation behavior of the system. The term  $\xi_1(x,t)u$  was added to model the effect of parametric noise or multiplicative noise and the term  $\xi_2(x,t)$  was added in [5] to model the effect of additive noise. Both terms  $\xi_1(x,t)u$  and  $\xi_2(x,t)$  represents Gaussian white noise and is assumed to be distributed with zero mean and uncorrelated over space and time.

#### 3 Integration of the KS equation

The numerical scheme used to integrate the KS equation is an extension of the work presented in [1]. This scheme uses the Distributed Approximating Functionals (DAFs) to achieve highly accurate spatial derivatives through a scheme based on the Crank-Nicolson method for the evolution over time. The domain is circular so that we can compare with related experiments conducted on circular burners. For convenience the KS equation is then converted to polar coordinates. In addition, the system is discretized in a grid with 32 point in the radial direction and 64 in the azimuthal direction (2048) total) (see Figure 1). In order to numerically solve the nonlinearity, the scheme employs Newton iteration in each time-step, in which the resulting sequence of linear systems is solved using the preconditioned Bi-CGSTAB method and when it fails to converge uses the LU decomposition to solve the aforementioned linear systems.



Figure 1: The Polar grid. The innermost points are located at a radius  $r = \frac{dr}{2}$  therefore the center is left out due to the singularity at r = 0. The radial spacing is dr. The size of our grid is 32 radial points, and 64 azimuthal points, i.e.  $dr = \frac{R}{32.5}$  and  $\phi = \frac{2\pi}{64}$ .

### 4 Numerical Challenges of the Integration

It is documented in [1] that the numerical investigation of the KS equation had been very difficult due to the singularity that arises in the biharmonic operator:  $\nabla^4 =$  $\left[\partial_{rr} + 2\left(\frac{1}{r}\right)\partial_r + \left(\frac{1}{r^2}\right)\partial_{\phi\phi}\right]^2$  near the origin of the polar grid. Although this singularity can be avoided by partitioning the each diameter into an even of equally spaced lattice points, as is shown in Figure 1, the presence of very small denominators at points close to origin such as in the term  $(\frac{1}{r^4})\partial_{\phi\phi\phi\phi}$ , make the resulting system extremely ill conditioned and sensitive to error in the spatial derivatives. This problem was mitigated by the use of both the DAFs approximation for the spatial derivatives and the scheme based on Crank-Nicolson method for the temporal approximation allowing long-term integration. However, the inclusion of the multiplicative noise term caused the integration to fail for some combinations of radius and noise intensity. After carefully reviewing the algorithm, we realized that the portion of the integration that was failing was the solution of the linear system using the Bi-CGSTAB method. We improve the integration by solving the system of equations using a direct method (LU decomposition) when the Bi-CGSTAB method fails.

# 5 Results of the integration of the KS Eqaution

We were able to successfully include a multiplicative noise term in the Kuramoto-Sivashinsky equation and modify the algorithm described in [1] and [5] to allow the integration of aforementioned equation. The parameters  $(\eta_1, \eta_2, \eta_3) = (0.32, 1.00, 0.017)$ were held constant whereas the radius of the burner and the intensity of the noise varied.

#### 5.1 Multiple ring transition

The transition between a single ring to multiple rings of cells happens for the range of radii from r = 8.50 to r = 9.00. Figure 2 shows a representative sample of two stationary states with a single ring and two stationary states with double-ring structure. We were able to uncover that the transition goes from 5 cells in a single ring to 5 cells in an outer ring and a single cell in an inner ring.

Intensity of Gaussian White Noise								
	0.00	0.20	0.25	0.30	0.35	0.40	0.45	0.50
7.7400	3 H	3H	3H	3R	3 R	3 R	3R	3 R
7.7405	3 H	3H	3H	3R	3 R	3 R	3R	3 R
7.7410	3H	3H	3H	3R	3 R	3 R	3R	3 R
7.7415	3 H	3H	3H	3 H	3 R	3 R	3R	3 R
7.7420	3 H	3H	3H	3 H	3 R	3 R	3R	3 R
7.7425	3H	3H	3H	3 H	3 R	3 R	3R	3 R
7.7430	3 H	3H	3H	3 H	3 R	3 R	3R	3 R
7.7435	3 H	3H	3H	3 H	3H	3 R	3R	3 R
7.7440	3 H	3H	3H	3 H	3H	3 R	3R	3 R
7.7445	3H	3H	3H	3 H	3H	3 R	3R	3 R
7.7450	3H	3H	3H	3 H	3H	3 R	3R	3 R
7.7455	3 H	3H	3H	3 H	3H	3 R	3R	3 R
7.7460	3 H	3H	3H	3 H	3H	3 H	3R	3 R
7.7465	3 H	3H	3H	3 H	3H	3 H	3R	3 R
7.7470	3 H	3H	3H	3 H	3H	3 H	3R	3 R
7.7475	3 H	3H	3H	3 H	3H	3 H	3R	3 R
7.7480	3 H	3H	3H	3 H	3H	3 H	3R	3 R
7.7485	3 H	3H	3H	3 H	3H	3 H	3R	3 R
7.7490	3 H	3H	3H	3 H	3H	3 H	3R	3 R
7.7495	3 H	3H	3H	3 H	3H	3 H	3R	3 R
3 R	3 Cells rotating							
3H	3 Cells hopping							

Figure 3: Behavior of the hopping pattern under multiplicative noise for different radii and noise intensities.

Figure 2: Multiple rings patterns found in the range from R = 8.50 to 9.00 and multiplicative noise intensity of 0.200.

5S Pattern Radius = 8.515

6-1S Pattern Radius = 8.530

#### 5.2 Hopping patterns

4S Pattern Radius = 8.500

5-1S Pattern Radius = 8 520

A hopping state can be described as a pattern in which individual cells sequentially make abrupt changes in their angular position while they keep rotating in a ring structure (see Figure 4). We focused on the range of radii from 7.7400 to 7.7495 becuase was in this interval where hopping cellular patterns were observed for the first time in a simulation [2]. We thought this was the logical next level of complexity to study. We found that as we increase the intensity of the multiplicative noise, the radius' interval where the hopping pattern is observed decreases and the cells tend toward a rigid rotation, where each of the 3 cells have different shape. We completed a study (see Figure 3) where the above statement is illustrated. Some of the results can be seen in Figure 4 where we show the time evolution of the pattern in the absence of noise and the time evolution of the pattern under multiplicative noise with a noise intensity of 0.35.



Figure 4: Evolution of the Hopping pattern. The top set of snapshots correspond to R = 7.7415 and noise intensity = 0.0 and the bottom set corresponds to R = 7.7415 and noise intensity = 0.35.

#### 6 Normal Forms

The center manifold theorem states that the full dynamics of a nonlinear model can be reduced to the center manifold near a bifurcation point. The dynamics on the center manifold is described by normal forms or amplitude equations, which are universal, i.e., all systems showing certain bifurcation have the same dynamics on the center manifold [4]. The amplitude equations can be expressed in terms of the parameters of the original system by devising a mapping to the center manifold of the wave vector space. This reduces the dimensionality of the problem from the dimension of the phase space (described by PDEs) to that of the center manifold (described by ODEs)[6]. In summary, the main idea of normal forms is the use near identity transformations that leads to a simpler differential equation which, close to equilibrium point, exhibits similar behavior as the original system. By taking advantage of the circular O(2) symmetry of the system, Palacios et. al. [3] used symmetry-based arguments to derive the normal form equations for studying the temporal behavior of spatiotemporal dynamic cellular pattern known as hopping state.

The results of the POD analysis are shown in Figure 5, which depicts the timeaverage (considered mode  $\Phi_0$ ) followed by the ten modes,  $\Phi_1$ - $\Phi_{10}$ , with the highest POD energy. The actual amount of energy in each mode is indicated below each graph. Each mode shows some amount of symmetry (see [3]). Hence, up to these subtle differences, it is reasonable to identify the Fourier-Bessel modes  $\Psi_{21}$ ,  $\Psi_{31}$ , and  $\Psi_{41}$ , as the principal modes of hopping states which ultimately let to the derivation of the set of equations found in[3], we added 6 new terms to the set of equations derived in [3] to account for multiplicative and additive noise:

$$\begin{aligned} \dot{z}_2 &= \bar{z}_2 z_4 + \alpha_2 z_3^2 \bar{z}_4 + z_2 (\xi_1 + \mu_2 + e_{22} |z_2|^2 \\ &+ e_{23} |z_3|^2 + e_{24} |z_4|^2) + \eta_1 \\ \dot{z}_3 &= \alpha_3 z_2 \bar{z}_3 z_4 + z_3 (\xi_2 + \mu_3 + e_{32} |z_2|^2 \\ &+ e_{33} |z_3|^2 + e_{34} |z_4|^2) + \eta_2 \\ \dot{z}_4 &= \pm z_2^2 + \alpha_4 z_3^2 \bar{z}_2 + z_4 (\xi_3 + \mu_4 + e_{42} |z_2|^2 \\ &+ e_{43} |z_3|^2 + e_{44} |z_4|^2) + \eta_3, \end{aligned}$$

$$(1)$$

where  $\alpha_2$ ,  $\alpha_3$ , and  $\alpha_4$ , are real-valued constants.  $\eta_1$ ,  $\eta_2$ ,  $\eta_3$ ,  $\xi_1$ ,  $\xi_2$  and  $\xi_3$  represent Gaussian white noise and are assumed to be distributed with zero mean and uncorrelated over space and time.



Figure 5: POD decomposition of a numerical three-cell hopping state, see Fig. 4, found in computer simulations of the KS model.

## 7 Results of the integration of the normal form equations

We were able to include a multiplicative noise term and an additive noise term in the normal form equations found in [3] and integrate those equations using the Euler-Maruyama method. We reproduced the results reported in [3] and compared them with the results obtained in the presence of multiplicative and additive noise. We show phasespace portraits as well as the probability density function (PDF) of the simulations involving noise functions. The parameters used are:  $\mu = (0.235, 0.2, 0.35), \alpha = (0, 2, 0),$  $e_2 = (-4, 0, -1), e_3 = (-1.6, -1, -1.6),$  $e_4 = (-2, 0, -2)$ . In this case the phasespace portraits and the PDF in the absence of noise (see figure 6) show that the solution is a modulated traveling wave. When either multiplicative (see Figure 7) or additive (see Figure 8) noise is added the modulations disappear and the trajectory seems to be, as expected, random.

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Figure 7: Phase-space portraits and Probability density function (PDF) with multiplicative noise. Noise intensity  $= 10^{-3}$ 



Figure 6: Phase-space portraits and Probability density function (PDF) without noise.



Figure 8: Phase-space portraits and Probability density function (PDF) with additive noise. Noise intensity =  $5 \times 10^{-5}$