# Azimuthal Modulational Instability of Vortices in the Nonlinear Schrödinger Equation

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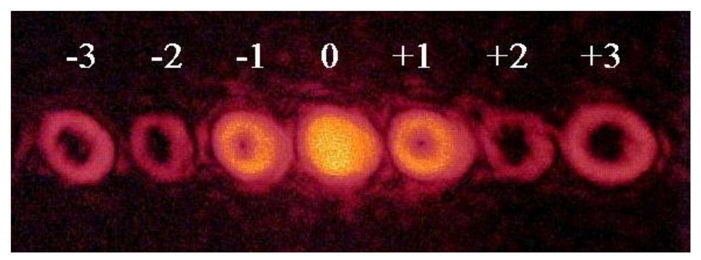
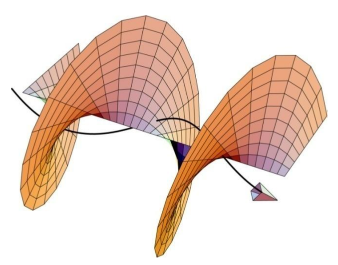
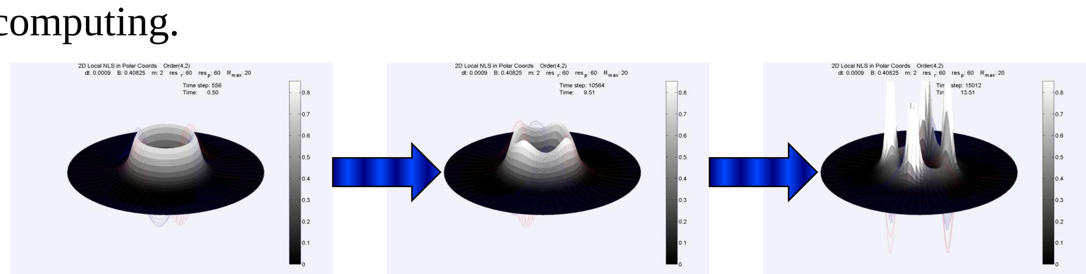
## BACKGROUND AND PURPOSE

The Nonlinear Schrödinger Equation (NLS) is used to describe various phenomena including Bose-Einstein Condensates (BECs), light propagation in nonlinear optical fibers and nonlinear crystals.



An interesting property of light is that of "twisted light". This is when light is twisted about its axis, and has a topological charge which tells how many times it twists around for each wavelength. Such light is called an optical vortex, and has the property that the center is identically zero in intensity, and hence looks like a "ring" of light. A higher charge vortex will result in more angular momentum, and thus a wider radius to the ring as shown above. When such vortices propagate through a nonlinear crystal, they exhibit azimuthal modulational instability (MI) as depicted below, and their dynamics can be described by the NLS. These vortices have potential applications in cryptography and quantum computing.



Azimuthal Modulational Instability

It would be useful to be able to understand the MI of such vortices. The method for this understanding should be general enough to be able to expand the ideas to include other nonlinearities and potentials in the NLS which would correspond to better physical models in this as well as other vortex solution applications such as in BECs. This project plans to do just that. In order to investigate the MI of the vortex solutions we set up our 2D problem in such a way that it becomes a quasi-1D problem, and then do a stability analysis in the fourier domain to predict the growth of the azimuthal modes. We find a steady state vortex solution by using numerical optimization, and then run full 2D simulations on a polar grid to test the predictions made by the theory.

## QUASI-1D AZIMUTHAL EQUATION OF MOTION

First, we start with the 2D NLS:

$$i\Psi_t + \nabla^2\Psi + s\,|\Psi|^2\Psi = 0$$

where the laplacian in polar coordinates is:

$$\nabla^2\Psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}$$

We can define the Action functional as:

$$S = \int_o^\infty L\,dt \qquad L = \int_0^{2\pi}\int_0^\infty \mathcal{L}\,r\,dr\,d\theta$$

where the Lagrangian density is:

$$\mathcal{L} = \frac{i}{2}\left(\Psi\Psi_t^* - \Psi^*\Psi_t\right) + \left|\Psi_r + \frac{1}{r}\Psi_\theta\right|^2 - \frac{s}{2}|\Psi|^4$$

Now, we assume a separable steady-state solution of the form:

$$\Psi(r,\theta,t) = f(r)\,A(\theta,t)$$

Plugging this solution into the Lagrangian, we have:

$$L = \int_0^{2\pi}\left(\frac{i}{2}C_1(AA_t^* - A^*A_t) + C_2|A|^2 + C_3|A_\theta|^2 + C_5A_\theta^*A + C_6A_\theta A^* - \frac{s}{2}C_4|A|^4\right)d\theta$$

where all the radial integrals become the following constants:

$$C_1 = \int_0^\infty |f(r)|^2\,r\,dr \qquad C_2 = \int_0^\infty \left|\tfrac{df}{dr}\right|^2 r\,dr$$
$$C_3 = \int_0^\infty \tfrac{1}{r^2}\,|f(r)|^2\,r\,dr \qquad C_4 = \int_0^\infty |f(r)|^4\,r\,dr$$
$$C_5 = \int_0^\infty \tfrac{1}{r}\tfrac{df}{dr}f(r)^*\,r\,dr \qquad C_6 = \int_0^\infty \tfrac{1}{r}\left(\tfrac{df}{dr}\right)^* f(r)\,r\,dr$$

We can now use the variational principle:

$$\frac{\delta S}{\delta A^\star} = \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\left[A_t^\star\right]} + \frac{\partial}{\partial\theta}\frac{\partial\mathcal{L}}{\partial\left[A_\theta^\star\right]} - \frac{\partial\mathcal{L}}{\partial A^\star} = 0$$

which leads us to a quasi-1D azimuthal equation of motion, which, if we perform the following rescaling

$$A \rightarrow A\,e^{\left(-i\frac{C_2}{C_1}t\right)} \qquad t \rightarrow \frac{C_3}{C_1}\,t$$

becomes:

$$\boxed{iA_t = -A_{\theta\theta} - s\frac{C_4}{C_3}|A|^2A}$$

## STABILITY ANALYSIS

For the stability analysis, we start by perturbing a plane-wave solution with a complex time-dependant perturbation of the form:

$$A(\theta,t) = \left(1 + u(\theta,t) + iv(\theta,t)\right)\,e^{i(m\theta+\Omega t)}$$

Plugging this into the A-equation, we get a pair of coupled PDEs describing the motion of the perturbation. To study the growth of azimuthal modes, we first expand u and v in a discrete Fourier series:

$$u(\theta,t) = \frac{1}{2\pi}\sum_{K=-\infty}^{\infty}\hat{u}(K,t)e^{-iK\theta} \qquad v(\theta,t) = \frac{1}{2\pi}\sum_{K=-\infty}^{\infty}\hat{v}(K,t)e^{-iK\theta}$$

where the amplitudes of each mode are given by the transforms:

$$\hat{u}(K,t) = \int_0^{2\pi}u(\theta,t)\,e^{iK\theta}\,d\theta \qquad \hat{v}(K,t) = \int_0^{2\pi}v(\theta,t)\,e^{iK\theta}\,d\theta$$

If we apply these transforms to the PDEs, and set ourselves on a rotating frame by a rescale of time, we arrive at a coupled pair of ODEs which govern the amplitude of each mode:

$$\hat{u}_t = K^2\hat{v} - \left[s\tfrac{C_4}{C_3}\left(2\hat{u}*\hat{v} + \hat{u}*\hat{u}*\hat{v} + \hat{v}*\hat{v}*\hat{v}\right)\right]$$

$$\hat{v}_t = \left(2s\tfrac{C_4}{C_3} - K^2\right)\hat{u} + \left[s\tfrac{C_4}{C_3}\left(\hat{v}*\hat{v} + 3\hat{u}*\hat{u} + \hat{v}*\hat{v}*\hat{u} + \hat{u}*\hat{u}*\hat{u}\right)\right]$$

If we linearize the system, we can put it into matrix form as:

$$\begin{bmatrix}\hat{u}_t\\ \hat{v}_t\end{bmatrix} = \begin{bmatrix}0 & K^2\\ \left(2s\frac{C_4}{C_3} - K^2\right) & 0\end{bmatrix}\begin{bmatrix}\hat{u}\\ \hat{v}\end{bmatrix}$$

The Eigenvalues (i.e. growth rates for each mode, with time rescaling taken into account) are:

$$\boxed{\lambda_{1/2} = \pm\frac{C_3}{C_1}\sqrt{K^2\left(K_{\text{crit}}^2 - K^2\right)}}$$

Where the critical mode (where all modes below it are unstable) is given by:

$$\boxed{K_{\text{crit}} = \pm\sqrt{2s\frac{C_4}{C_3}}}$$

## NUMERICAL METHOD FOR STEADY-STATE SOLUTION

To obtain the radial profile of a steady-state vortex solution, we first plug in a separable ansatz into the NLS to get:

$$\Psi(r,\theta,t) = f(r)\,e^{im\theta}\,e^{i\Omega t}$$

$$\vec{F}(f(r)) = -\left(\Omega + \frac{m^2}{r^2}\right)f(r) + \frac{1}{r}f'(r) + f''(r) + s\,|f(r)|^2 f(r) = 0$$

We then use a modified Gauss-Newton optimization algorithm:

We define a Merit Function: $M(\vec{f}) = \frac{1}{2}\sum_{i=1}^{n}(F_i(\vec{f}))^2$

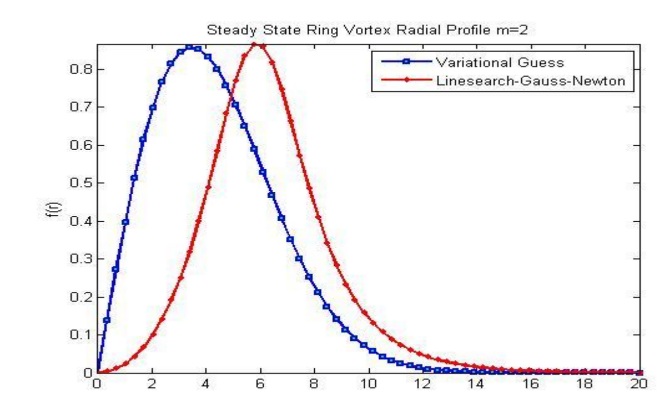
Then we can iterate our solution as: $\vec{f}_{k+1} = \vec{f}_k + \alpha_k\vec{p}_k$

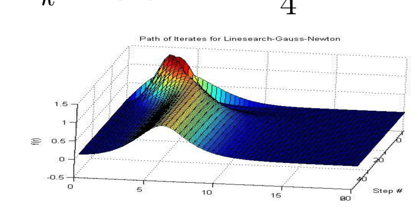
where the step direction is: $\vec{p}_k = -(J_k^TJ_k + \lambda_kI)^{-1}J_k^T\vec{f}_k$

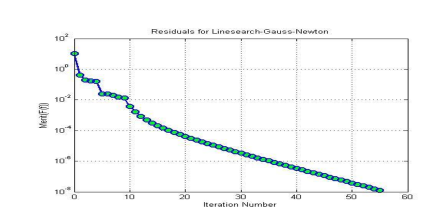
and the step length is: $\min_{\alpha>0} M(\vec{f}_k + \alpha\vec{p}_k) \rightarrow \alpha_k$

For an example case, we use the initial guess: $\vec{f}_0 = B\,r\exp\left(\frac{-r^2}{2\sigma^2}\right)$

We get the following result in 55 steps: $\lambda_k = 0.01 \quad \Omega = \frac{1}{4} \quad m = 2$




## NUMERICAL METHOD FOR FULL 2D SIMULATION

We simulate the system on a polar grid using finite difference. For the time derivative, we use the 4th order accurate Runge-Kutta:

$$\Psi_t = F(\Psi) = i(\nabla^2\Psi + s|\Psi|^2\Psi)$$

$$\Psi^{n+1} = \Psi^n + \frac{\Delta t}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$k_1 = F(\Psi^n) \qquad k_2 = F(\Psi^n + \frac{\Delta t}{2}k_1)$$

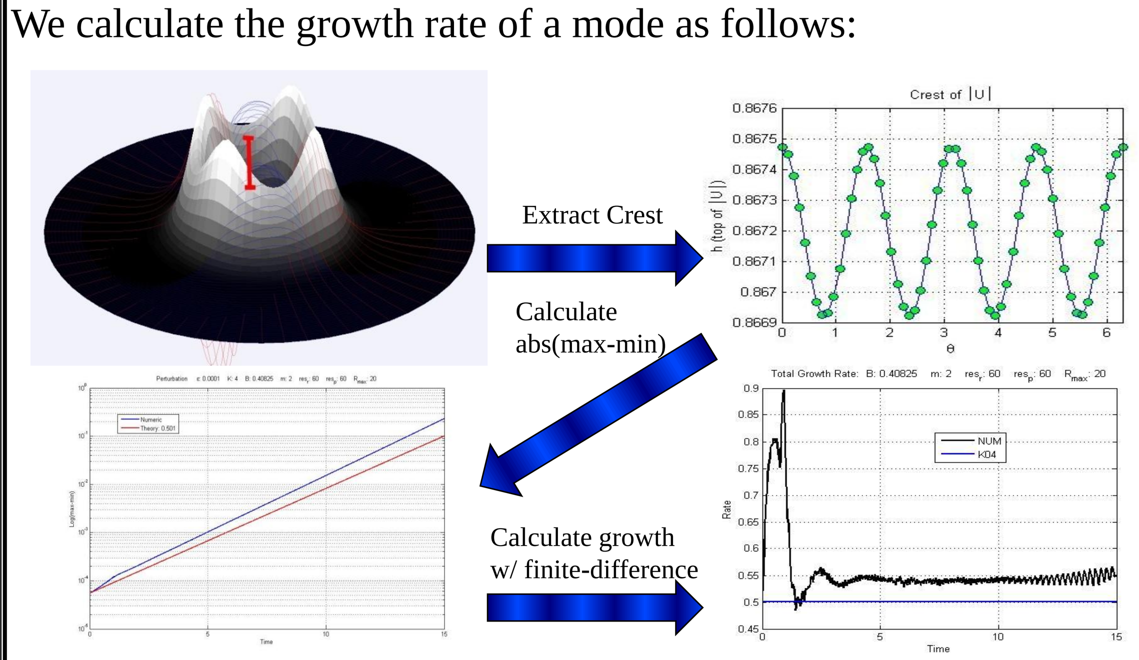
$$k_3 = F(\Psi^n + \frac{\Delta t}{2}k_2) \qquad k_4 = F(\Psi^n + \Delta t k_3)$$

combined with the following 2nd order differencing in space:

$$F(\Psi_{i,j}) = i(\nabla^2\Psi_{i,j} + s|\Psi_{i,j}|^2\Psi_{i,j})$$

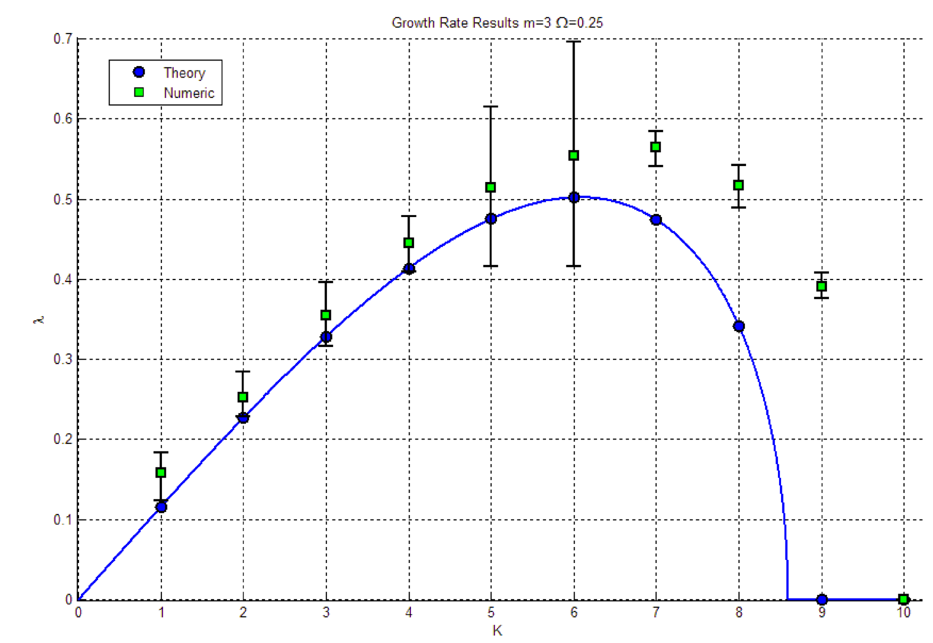
$$\nabla^2\Psi_{i,j} = \frac{1}{r_i}\frac{1}{\Delta r}\left(r_{i+\frac{1}{2}}\frac{\Psi_{i+1,j} - \Psi_{i,j}}{\Delta r} - r_{i-\frac{1}{2}}\frac{\Psi_{i,j} - \Psi_{i-1,j}}{\Delta r}\right) + \frac{1}{r_i^2}\frac{\Psi_{i,j+1} - 2\Psi_{i,j} + \Psi_{i,j-1}}{\Delta\theta^2}$$

We calculate the growth rate of a mode as follows:



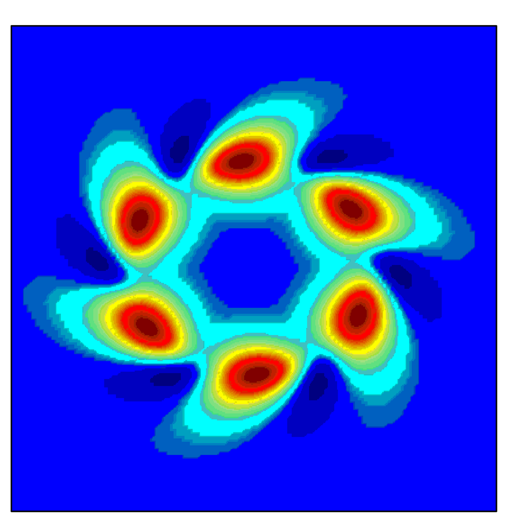

## THEORETICAL PREDICTIONS AND NUMERICAL RESULTS

As an example case, we set m=3 and run the simulations. The following is a plot of the eigenvalues/growth-rates that the theory predicts plotted with the numerical results for each mode:




## CONCLUSIONS

The results are very close to those predicted. We believe that the small discrepancy is due to the fact that we assumed a purely separable solution, which is a good approximation of what the full 2D modes are, but in reality (as can be seen in the figure below) the modes have some radial-azimuthal coupling, which seems to be having an effect on the growth rates, making them higher than predicted.



That being said, the method described here does appear to be quite useful in approximately describing and predicting the MI of vortices. Thus, the technique used here can be applied to nonlocal, saturation, or other alternative nonlinearities, as well as problems including an external potential.