Azimuthal Modulational Instability of Vortices in the Nonlinear Schrödinger Equation

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BACKGROUND AND PURPOSE

The Nonlinear Schrödinger Equation (NLS) is used to describe various phenomena including Bose-Einstein Condensates (BECs), light propagation in nonlinear optical fibers and nonlinear crystals.

An interesting property of light is that of “twisted light”. This is when light is twisted about its axis, and has a topological charge which tells how many times it twists around for each wavelength. Such light is called an optical vortex, and has the property that its center is identically zero in intensity, and hence looks like a “ring” of light. A higher charge vortex will result in more angular momentum, and thus a wider radius to the ring as shown above. Such light is called an optical vortex, and has the property that the center is identically zero in intensity, and hence looks like a “ring” of light. When such vortices propagate through a nonlinear crystal, they exhibit azimuthal modulational instability (MI) as depicted below. Thus, the technique used here can be applied to various phenomena including an external potential.

NUMERICAL METHOD FOR STEADY-STATE SOLUTION

Now, we assume a separable steady-state solution of the form:

\[ \Psi(x, \theta, t) = \psi(x) \phi(\theta) \]

Plugging this into the Lagrangian, we have:

\[ L = \int \left[ \frac{1}{2} \left( \psi_x^2 + \psi_y^2 + \psi^{(2)} \right) + \frac{1}{2} \left( \phi_x^2 + \phi_y^2 + \phi^{(2)} \right) \right] dx \]

where the Lagrangian density is:

\[ L = \frac{1}{2} \left( \psi_x^2 + \psi_y^2 + \psi^{(2)} \right) + \frac{1}{2} \left( \phi_x^2 + \phi_y^2 + \phi^{(2)} \right) \]

and the system, we can put it into matrix form as:

\[ \mathbf{A} \mathbf{u} = \mathbf{0} \]

where \( \mathbf{A} \) is a matrix and \( \mathbf{u} \) is a vector.

For the stability analysis, we start by perturbing a plane-wave solution with a complex time-dependent perturbation of the form:

\[ \Psi(x, \theta, t) = (1 + u(x, \theta, t) + iv(x, \theta, t)) e^{i(m\theta + \omega t)} \]

Plugging this into the NLS equation, we get a pair of coupled PDEs describing the motion of the perturbation. To study the growth of azimuthal modes, we first expand \( u \) and \( v \) in a discrete Fourier series:

\[ u(K) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x, \theta) e^{-iK\theta} d\theta \]

where \( u(K) \) is the Fourier coefficient of \( u \) at wavenumber \( K \).

If we apply these transforms to the PDEs, and set ourselves on a rotating frame by a rescale of time, we arrive at a coupled pair of ODEs which govern the amplitude of each mode:

\[ \dot{u}(K) = -i \left( \frac{K^2}{2} + \omega \right) u(K) \]

where the time derivative is:

\[ \dot{u} = \frac{1}{i} \left( \frac{K^2}{2} + \omega \right) u \]

If we linearize the system, we can put it into matrix form as:

\[ \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} -i \left( \frac{K^2}{2} + \omega \right) & 0 \\ 0 & -i \left( \frac{K^2}{2} + \omega \right) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \]

The Eigenvalues (i.e. growth rates for each mode, with time rescaling taken into account) are:

\[ \lambda_{1,2} = \pm \frac{C_3}{C_4} \sqrt{K^2 (K_{crit}^2 - K^2)} \]

where the critical mode (where all modes below it are unstable) is given by:

\[ K_{crit} = \pm \frac{C_3}{C_4} \sqrt{2} \]

STABILITY ANALYSIS

QUASI-1D AZIMUTHAL EQUATION OF MOTION

First, we start with the 2D NLS:

\[ i \psi_t + \psi_{xx} + \psi_{yy} + \psi^{(2)} = 0 \]

where the Laplacian in polar coordinates is:

\[ \nabla^2 \psi = \frac{1}{r} \left( \frac{1}{r} \psi_r^2 + \psi_r \right) \]

We can define the Action functional as:

\[ S = \int_0^L \int_0^L \left[ \frac{1}{2} \left( \psi_x^2 + \psi_y^2 + \psi^{(2)} \right) + \frac{1}{2} \left( \phi_x^2 + \phi_y^2 + \phi^{(2)} \right) \right] dx \]

where the Lagrangian density is:

\[ \mathcal{L} = \frac{1}{2} \left( \psi_x^2 + \psi_y^2 + \psi^{(2)} \right) + \frac{1}{2} \left( \phi_x^2 + \phi_y^2 + \phi^{(2)} \right) \]

NUMERICAL METHOD FOR FULL 2D SIMULATION

To obtain the radial profile of a steady-state vortex solution, we first plug in a separable ansatz into the NLS to get:

\[ \Psi(r, \theta, t) = e^{i(m\theta + \omega t)} \]

We then use a modified Gauss-Newton optimization algorithm:

\[ \min_{\theta} \sum_j \left( \frac{1}{2} \right) \left( \frac{m^2}{2} \right) \left( \frac{m}{2} \right) \]

which leads us to a quasi-1D azimuthal equation of motion, which, if we perform the following rescaling:

\[ A \rightarrow A e^{-i\theta \theta} \quad \xi = \frac{C_3}{C_4} \]

becomes:

\[ \dot{A}_1 = -A_0 \quad \dot{A}_2 = -A_2 \]

As an example case, we use the initial guess:

\[ f_0 = B r e^{i (m \theta - \theta)} \]

We get the following result in 55 steps:

\[ \lambda_{1,2} = 0.01 \quad m = 2 \]

NUMERICAL PREDICTIONS AND NUMERICAL RESULTS

CONCLUSIONS

The results are very close to those predicted. We believe that the small discrepancy is due to the fact that we assumed a purely separable solution, which is a good approximation of what the full 2D modes are, but in reality (as can be seen in the figure below) the modes have some radial-azimuthal coupling, which seems to be having an effect on the growth rates, making them higher than predicted.

That being said, the method described here does appear to be quite useful in approximately describing and predicting the MI of vortices. Thus, the technique used here can be applied to nonlocal, saturation, or other alternative nonlinearities, as well as problems including an external potential.